# Tensor Products and Kirchberg's Conjectures

## Seminar[1] – Summer Semester 2023

### J.Prof. Dr. Kristin Courtney and AR Dr. Sam Evington

Talks are assigned on a first-come-first-serve basis. Contact one of the organizers in person or by email to request a particular talk. (Later dates are subject to possible change depending on speaker schedules.)

You should arrange a meeting with one or both of the organizers to discuss a general outline as well as the treatment of supplementary/peripheral material before you begin preparing your lecture. A draft of the lecture notes should be submitted at least two weeks before the presentation, at which point a meeting with one or both of the organizers should be arranged to discuss the presentation and clarify any details from the material as needed. Written lecture notes should accompany any presentation that is to be graded.
**Please notify us in advance if you wish to receive a grade for the presentation.**

A summary of the upshots of lectures 1-9 will be regularly updated here.

**1. Fundamentals for tensor products of C\*-algebras.**
Speaker: Kristin Courtney
Date: 04.05.23
Contents: Important facts from tensor products of complex algebras and Hilbert space operators, highlights from maximal and spatial tensor products, tensor products and cp maps, injectivity and Arveson's theorem, operator systems.
Literature: [5, §3.1-3.3].

**2. (Weak) Injectivity.**
Speaker: Lukas Obermeyer
Date: 11.04.23
Contents: Lance's Weak Expectation Property (WEP), inclusions of tensor products, and max tensor of a nuclear map with identity factors through min tensor product.
Sources: [5, §3.6].

**3. Exact Sequences.**
Speaker: Vishnu Sankarankutty
Date: 18.04.23
Contents: Exactness of the functors $(\cdot) \otimes_{min} A$ and $(\cdot) \otimes_{max} A$ for a fixed C\*-algebra $A$, characterize nuclearity of maps via tensor products with $C^*(\mathbb{F})$.
Sources:[5, §3.7 up to pg. 96 (possibly omitting pg. 94), Corollary 3.8.8.]

**4. Unique C\*-tensor norm on $B(\mathcal{H}) \otimes C^*(\mathbb{F})$ and consequences**
Speaker: Sam Evington
Date: 25.04.23
Contents: Unique C\*-tensor norm on $B(\mathcal{H}) \otimes C^*(\mathbb{F})$ (via Pisier's free product strengthening), tensorial characterizations of LLP and WEP via $B(\mathcal{H})$ and $C^*(\mathbb{F})$, and nuclear embeddability $\Longleftrightarrow$ exactness.
Sources: [19, Proposition 8.8, Chapter 13], [20, Chapter 9], [5, §13.2].

**5. Kirchberg's Local Lifting Property.**
Speaker: Kristin Courtney
Date: 02.05.23
Contents: (Local) lifting property (LLP), LLP for nuclear C\*-algebras, LLP for $C^*(\mathbb{F}_n)$ (using Kasparov's Stinespring Dilation theorem and the noncommutative Tietze Extension theorem), Choi-Effros

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[1]The seminar is aimed at masters students who have seen Operator Algebras I. It can be used to fulfil a Specialization Supplement and Research Skills module (Ma-E) or a Specialization Module (Ma-S2 (Geometric Structures) or Ma-S3 (Operator Algebras and Noncommutative Geometry)).

and Effros-Haagerup lifting theorems.
Sources: [5, §13.1, Appendix C], [16, §3], [15, §2].

## 6. Connes Embedding Problem
Speaker: Ennes Kurt
Date: 09.05.23
Contents: Introduce ultralimits/ ultrapowers of von Neumann algebras (including the hyperfinite $II_1$ factor), Connes Embedding Problem, hyperlinearity, and matrix-microstates/ random matrix formulation.
Sources: [9, §2.1-2.3, 3.6, 3.7], [20, §11.1, 11.5, 12.1-12.5], [18, §1], [22, §2.2], [7, §3].

## 7. Tensor Products of RF group C\*-algebras and RFD C\*-algebras.
Speaker: Akshara Prasad
Date:16.05.23
Contents: Exactness and amenability for residually finite groups, residual finite dimensionality for $C^*(\mathbb{F}_2)$, $C^*(\mathbb{F}_2 \times \mathbb{F}_2) \cong C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2)$, maximal tensor product of two RFD C\*-algebras is RFD iff there is a unique tensor norm, faithful traces on separable RFD C\*-algebras.
Sources: [5, §3.7 from pg.96], [6, Theorem 7, Corollary 9], [16, Proposition 3.19 (i)-(iii), Lemma 3.20], [20, §9.2], ([2, Prop 2.1, Lemma 3.2]).

## 8. Kirchberg's Conjectures
Speaker: Kristin Courtney
Date: 23.05.23
Contents: QWEP, permanence properties for (Q)WEP, . Kirchberg's QWEP conjecture, characterization via tensors on $C^*(\mathbb{F})$, proof that CEP is equivalent to QWEP and to $C^*(\mathbb{F})$ WEP.
Sources: [5, §13.3 not including CEP], [16, Proposition 3.19], [15, §8], [20, §9.7].

## 9. NO SEMINAR THIS WEEK
Date: 06.06.23

## 10. NO SEMINAR THIS WEEK
Date: 13.06.23

## 11. Tsirleson's Problem
Speaker: Andrea Vaccaro
Date: 20.06.23
Contents: Quantum (Commuting) Correlation sets/matrices/unitaries, universal C\*-algebras of projection valued measures and QWEP (going from $C^*(\mathbb{F})$ to $C^*(\mathbb{Z}_m^{*k})$).
Sources: [20, Chapter 16], [8, §3], [14, §5], [22, §2.2], [18, §2].

## 10. NO SEMINAR THIS WEEK
Date: 27.06.23

## 13. Group C\*-algebras without the LLP
Speaker: Kristin Courtney
Date: 05.07.23 at 14:15 (Zoom Seminar)
Contents: Thom's non-residually finite hyperlinear property (T) group: Kirchberg's factorization property and residual finiteness under property (T); $C^*(SL_3(\mathbb{Z}))$.
Sources: [5, §6.4], [1, §2.5, 4] [20, Chapter 17], [21, § 2], [13, §1, 2.2, 2.3, 3.1].

All sources should be available electronically and/or through the library. If you cannot access some of the source material, please let us know.

# References

[1] C. Anantharaman-Delaroche. "On tensor products of group C*-algebras and related topics." *Reseach Semester at the Bernoulli Center, EPFL,* Jan 2007, Lausanne, Switzerland. hal-00360401. `https://hal.science/hal-00360401/document`

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[3] N. P. Brown. "Connes' embedding problem and Lance's WEP." *Int. Math. Res. Not.* 2004, no. 10, 501-510. `https://arxiv.org/pdf/math/0305167.pdf`

[4] N. P. Brown. "Invariant means and finite representation theory of C*-algebras." *Mem. Amer. Math. Soc.* 184(865):viii+105, 2006.

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[6] M. Choi. "The full C*-algebra of the free group on two generators." *Pacific J. Math.*, 87(1):41–48, 1980. `https://msp.org/pjm/1980/87-1/pjm-v87-n1-p04-s.pdf`

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[11] I. Goldbring. "Model Theroy and the QWEP conjecture." *Illinois J. Math.*, 59(3):795-799, 2015. `https://www.math.uci.edu/~isaac/QWEPjournal.pdf`

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[16] N. Ozawa. "About the QWEP conjecture". *Internat. J. Math.*, 15(5):501-530, 2004. `https://arxiv.org/abs/math/0306067`

[17] V. Paulsen. *Completely bounded maps and operator algebras*, volume 78 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002.

[18] V. Paulsen. Lecture Notes on CEP from Copenhagen 2022 workshop. (many typos) `https://qmath.ku.dk/events/conferences/quantum-entanglement/Paulsen_-_CEP.pdf`

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[20] G. Pisier. *Tensor products of* C*-*algebras and operator spaces—the Connes-Kirchberg problem*, volume 96 of LMS Student Texts. Cambridge University Press, 2020. `https://www.math.tamu.edu/~pisier/TPCOS.pdf`

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