Tsirleson's Problem & Connes' Embedding Problem

Tsirleson's Problem asks whether two different mathematical models, both representing two observers making measurements in an independent manner, are equivalent or not.

In Quantum Mechanics, a system is represented by a Hilbert space H, and a measurement z with $m \in \mathbb{N}$ possible outcomes is described by a collection of projections $P_1, \dots, P_m \in B(H)$ such that:

1) $P_i \perp P_j \quad \forall i < j \leq m$

2) $\sum_{i \leq m} P_i = Id$

[We also refer to this as a PVM, a Projective Valued Measure, seen as a probability measure over $\{1, \dots, m\}$ with projections as values.]

In operator algebraic form, this is easily represented by a unital $*$-homomorphism

$$\Phi : \ell^m_\infty \to B(H)$$

A physical state is represented by a state on $B(H)$. Given a state, the probability of a given outcome $k \leq m$, is computed via a (vector) state ω on $B(H)$ as

$$p(z|k) = \omega(P_k) \quad \left(= \langle \xi, P_k \xi \rangle\right)$$

In the presence of two observers, the independence of the measurements is classically represented with commutation of the values of the corresponding PVMs.

Non-Relativistic Quantum Mechanics

Observer A measures in system H_A the event $\{P_1, \dots, P_m\} \subseteq B(H_A)$.

Algebraic Quantum Field Theory

Observers A and B measure in the same system H with

Observer B measures in system H_B the event $\{Q_1, \ldots, Q_m\} \subseteq B(H_A)$.

For a vector state $\xi \in H_A \otimes H_B$, the probability of the event (i,j) is

$$x_\xi(i,j) = \langle \xi, (P_i \otimes Q_j)\xi \rangle \in \mathbb{R}$$

We consider the more general case where A and B make $d \in \mathbb{N}$ measurements, hence we have $\{P_i^k\}_{i \leq m, k \leq d} \subseteq B(H_A)$, $\{Q_j^\ell\}_{j \leq m, \ell \leq d} \subseteq B(H_B)$

The set of quantum correlations records all possible combinations of probabilities of these events

$$Q_\otimes(m,d) = \{x_\xi \in \mathbb{R}^{k^2 m^2} \mid \xi \in H_A \otimes H_B, \text{ arbitrary } H_A \text{ and } H_B, \{P_i^k\} \subseteq B(H_A), \{Q_j^\ell\} \subseteq B(H_B)\}$$

where $x_\xi(i,k;j,\ell) = \langle \xi, (P_i^k \otimes Q_j^\ell)\xi \rangle$

$\{P_i^k\}_{i \leq m, k \leq d}, \{Q_j^\ell\}_{j \leq m, \ell \leq d} \subseteq B(H)$ such that $[P_i^k, Q_j^\ell] = 0 \quad \forall i,j,k,\ell$.

For a vector state $\xi \in H$, the probability of the event (i,j) is

$$\langle \xi, P_i^k Q_j^\ell \xi \rangle \in \mathbb{R}$$

and in this case the set of quantum correlations is

$$Q_c(m,d) = \{x_\xi \in \mathbb{R}^{k^2 m^2} \mid \xi \in H\}$$
for arbitrary H

where $x_\xi(i,k;j,\ell) = \langle \xi, P_i^k Q_j^\ell \xi \rangle$

Fact 1: $Q_c(m,d)$ is closed.

Pick $x_{\xi_\gamma} = (a_{ij}^{k\ell}(\gamma))$, with $\xi_\gamma \in H_\gamma$ inducing it and let $a = (a_{ij}^{k\ell})$ be the limit as $\gamma \to \infty$.

Fix $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$, take the Hilbert space ultraproduct $H(\mathcal{U}) = \prod_{\mathcal{U}} H(\gamma)$ and $\xi_{\mathcal{U}} = (\xi_\gamma)_{\gamma \in \Gamma}$, with projections $P_i^k(\mathcal{U}) = (P_i^k(\gamma))$, $Q_j^\ell(\mathcal{U}) = (Q_j^\ell(\gamma))$ will be s.t.

$$a_{ij}^{k\ell} = \mathcal{U}\text{-}\lim \langle \xi_\gamma, P_i^k(\gamma) Q_j^\ell(\gamma) \xi_\gamma \rangle = \langle \xi_{\mathcal{U}}, P_i^k(\mathcal{U}) Q_j^\ell(\mathcal{U}) \xi_{\mathcal{U}} \rangle$$

Fact 2: $\overline{Q_\otimes(m,d)} \subseteq Q_c(m,d)$

Tsirleson's Problem: Is $\overline{Q_\otimes(m,d)} = Q_c(m,d)$? (hard for $d > 2$)

Tsirleson's Problem has also a somewhat different interpretation.

Let $Q_s(m,d)$ be defined as $Q_c(m,d)$ but restricting to finite-dimensional H.

Then $\overline{Q_s(m,d)} = \overline{Q_\otimes(m,d)}$. [we prove this below]

How do we get from Kirchberg's Conjecture to Tsirleson's Problem?

Denote $A(m,d) = C^*(\underbrace{\mathbb{Z}/m\mathbb{Z} * \_\_ * \mathbb{Z}/m\mathbb{Z}}_{d \text{ times}}) \cong \ell_\infty^m * \_ * \ell_\infty^m$

Proposition 1. Let F be a state on $A(m,d) \otimes_{max} A(m,d)$, and let e_j^k be the j-th basis vector in the k-th copy of ℓ_∞^m in $A(m,d)$.

Set $x_F(i,k;j,\ell) = F(e_i^k \otimes e_j^\ell)$, then

$$Q_c(m,d) = \{x_F \in \mathbb{R}^{m^2d^2} \mid F \text{ is a state on } A(m,d) \otimes_{max} A(m,d)\}$$

Proposition 2. Let f be a state on $A(m,d) \otimes_{min} A(m,d)$, and let e_j^k be the j-th basis vector in the k-th copy of ℓ_∞^m in $A(m,d)$.

Set $x_f(i,k;j,\ell) = f(e_i^k \otimes e_j^\ell)$, then

$$\overline{Q_\otimes(m,d)} = \overline{Q_s(m,d)} = \{x_f \in \mathbb{R}^{m^2d^2} \mid f \text{ is a state on } A(m,d) \otimes_{min} A(m,d)\} =: S$$

Proposition 3.

$$C^*(\mathbb{F}_2) \otimes_{min} C^*(\mathbb{F}_2) \cong C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2) \iff A(m,d) \otimes_{min} A(m,d) \cong A(m,d) \otimes_{max} A(m,d)$$

Corollary [Fritz; Jung-Navascues-Palazuelos-Peréz-García-Scholz-Werner]

Kirchberg's Conjecture $\Rightarrow$ Tsirleson's Problem.

[$\Leftarrow$ is also true and was proved by Ozawa]

Proof of Proposition 1.

Tuples $\{P_i^k\}_{i\le m, k\le d}$, $\{Q_j^\ell\}_{j\le m, \ell\le d}$ s.t. $[P_i^k, Q_j^\ell]=0$ in $B(H)$ give unital $*$-homomorphisms

$$\Phi : A(m,d) \otimes_{max} A(m,d) \to B(H)$$
$$e_i^k \otimes e_j^\ell \mapsto P_i^k Q_j^\ell$$

So $Q_c(m,d) \subseteq \{x_F \in \mathbb{R}^{m^2d^2} \mid F \text{ state on } A(m,d) \otimes_{max} A(m,d)\}$ since every vector state on $B(H)$ induces a state on $A(m,d) \otimes_{max} A(m,d)$, while $\supseteq$ can be proved using the GNS representation corresponding to the given state F. $\square$

Proof of Proposition 2 Let S denote the right-hand side set in the statement.

$\boxed{\overline{Q_{\otimes}(m,d)} \subseteq S.}$ Every $x_\xi \in Q_{\otimes}(m,d)$, with $\xi \in H_A \otimes H_B$, $\{P_i^k\}_{i \le m, k \le d} \subseteq B(H_A)$ and $\{Q_j^\ell\}_{j \le m, \ell \le d} \subseteq B(H_B)$ naturally induces a representation

$$\Phi : A(m,d) \otimes_{min} A(m,d) \to B(H_A \otimes H_B)$$
$$e_i^k \otimes e_j^\ell \mapsto P_i^k \otimes Q_j^\ell$$

so ξ gives rise to a state on $A(m,d) \otimes_{min} A(m,d)$. Since S is <u>closed</u> (it is continuous image of the state space, which is compact), we have $\overline{Q_{\otimes}(m,d)} \subseteq S$.

$\boxed{\overline{Q_s(m,d)} \subseteq \overline{Q_{\otimes}(m,d)}.}$ Let H be fin. dim, $\xi \in H$, $\{P_i^k\}_{i \le m, k \le d}$, $\{Q_j^\ell\}_{j \le m, \ell \le d} \subseteq B(H)$.

This gives a $*$-homomorphism $\pi : A(m,d) \otimes_{alg} A(m,d) \to B(H)$ onto $C^*(A, B)$, with $A = C^*(\{P_i^k\}_{i \le m, k \le d})$ and $B = C^*(\{Q_j^\ell\}_{j \le m, \ell \le d})$. A and B are fin. dim, hence $A \otimes_{max} B = A \otimes_{min} B \subseteq B(H \otimes H)$. The state induced by ξ on $A \otimes_{min} B$ extends to a state ψ on $B(H \otimes H)$, hence there are $\xi_1, \dots, \xi_m \in H \otimes H$ and $\omega_1, \dots, \omega_m \ge 0$ with $\sum \omega_h = 1$ s.t.

$$\psi(x \otimes y) = \sum_{h \le m} \omega_h \langle \xi_h, (x \otimes y) \xi_h \rangle$$

We conclude with a standard multiplicity trick: let $K = \ell_2^m \otimes (H \otimes H)$ and consider the vector $\xi' = \sum_h \sqrt{\omega_h}\, e_h \otimes \xi_h \in K$.

Then

$$\psi(x \otimes y) = \sum_{h \le m} \langle \sqrt{\omega_h}\, \xi_h, (x \otimes y) \sqrt{\omega_h}\, \xi_h \rangle =$$
$$= \langle \sum \sqrt{\omega_h}\, e_h \otimes \xi_h, (1 \otimes (x \otimes y)) \sum \sqrt{\omega_h}\, e_h \otimes \xi_h \rangle = \langle \xi', (x' \otimes y') \xi' \rangle$$

where $x' = 1 \otimes x$ and $y' = y$ $\square$

We need some more lemmas to prove $S \subseteq \overline{Q_s(m,d)}$.

Def.

Given two C^*-algebras A_1, A_2, a linear functional φ on $A_1 \otimes_{alg} A_2 \subseteq B(H)$ <u>comes from a matricial vector state</u> if there are $n_1, n_2 \in \mathbb{N}$ and cp maps $u_i : A_i \to M_{n_i}$ and a vector state ψ on $M_{n_1} \otimes_{min} M_{n_2}$ s.t.

$$\varphi(x) = \psi\big(u_1 \otimes u_2(x)\big) \quad \forall x \in A_1 \otimes_{alg} A_2$$

Lemma 1.

Let $A_j \subseteq B(H_j)$ be unital C^*-algebras, for $j=1,2$. Let $\varphi : A_1 \otimes_{alg} A_2 \to \mathbb{C}$ be a linear form s.t. $\varphi(1 \otimes 1) = 1$. TFAE

1) φ extends to a state on $A_1 \otimes_{min} A_2$

2) φ is pointwise limit on $A_1 \otimes_{alg} A_2$ of a net of functionals that come from a matricial vector state.

Lemma 2.

Let $(Q_j)_{j \le m}$ be a PVM on a Hilbert space $\hat{H}$, and let $H \subseteq \hat{H}$ be fin. dim. There is a fin. dim K_0 with $H \subseteq K_0 \subseteq \hat{H}$ s.t. $\forall$ fin dim. K with $K_0 \subseteq K \subseteq \hat{H}$ there exists a PVM $(Q_j')_{j \le m}$ on K s.t. $P_H Q_j'|_H = P_H Q_j|_H \quad \forall j \le m$.

Proof.

Set $E_j = Q_j(H)$. Then P_{E_j} and Q_j commute and they are the same on H. Take $K_0 = E_1 + \dots + E_m$, and suppose $K_0 \subseteq K \subseteq \hat{H}$. [note that $K_0 \supseteq H$ since $\sum Q_j = 1$]

Let $Q_j' = P_{E_j} P_K$ for every $j < m$ and let $Q_m' = P_{E_m} P_K + \big(Id_K - (\sum_{j<m} Q_j')\big)$.

Given $\xi \in H$ we get $Q_j' \xi = P_{E_j} \xi = Q_j \xi$ if $j < m$ and

$$Q_m' \xi \in Q_m \xi + K \ominus K_0 \subseteq Q_m \xi + H^\perp, \text{ as desired.} \quad \square$$

Let $E(m,d) \subset A(m,d)$ be the linear subspace of all sums $x_1 + \dots + x_d$ where x_j belongs to the j-th copy of ℓ_∞^m in $A(m,d)$.

Lemma 3.

Let H_1, H_2 be finite dimensional and $u : A(m,d) \to B(H_1)$, $v : A(m,d) \to B(H_2)$ be ucp maps. There are fin. dim K_1, K_2 with $K_i \supseteq H_i$ and $*$-homomorphisms $\pi_i : A(m,d) \to B(K_i)$ s.t.

$$\forall x \in E(m,d) \otimes_{alg} E(m,d) \quad (u \otimes v)(x) = P_{H_1 \otimes H_2} (\pi_1 \otimes \pi_2)(x)|_{H_1 \otimes H_2}$$

Proof.

By Stinespring's theorem we can find $H_1, \hat{H}_2$ and unital $*$-homomorphisms $\hat{\pi}_i : A(m,d) \to B(\hat{H}_i)$ $i=1,2$ satisfying the statement of the lemma.

Let P^k (and Q^ℓ) be the PVMs corresponding to the k-th (ℓ-th) copy of ℓ^∞_m in $A(m,d)$, obtained through $\hat{\pi}_1$ ($\hat{\pi}_2$). By applying multiple times Lemma 2, we can find fin. dim K_1, K_2 with $H_i \subseteq K_i \subseteq \hat{H}_i$ and $(\tilde{P}^k_i)_{i\le m}$ $(\tilde{Q}^\ell_j)_{j\le m}$ s.t.

$$P_{H_1\otimes H_2} \cdot (\tilde{P}^k_i \otimes \tilde{Q}^\ell_j)_{|H_1\otimes H_2} = P_{H_1\otimes H_2} \cdot (P^k_i \otimes Q^\ell_j)_{|H_1\otimes H_2}$$

Then π_1, π_2 are obtained by mapping $A(m,d)$ to the PVMs $(\hat{P}^k_i)_{i\le m, k\le d}$ and $(\tilde{Q}^\ell_j)_{j\le m, \ell\le d}$. □

Back to the Proof of Proposition 2

$\boxed{S \subseteq \overline{Q_s(m,d)}}$ Take a state f on $A(m,d)\otimes_{\min} A(m,d)$. By Lemma 1 it is pointwise limit of a net of states $(F_\gamma)_\gamma$ with

$$F_\gamma(x\otimes y) = \psi_\gamma(u_1^\gamma(x)\otimes u_2^\gamma(y))$$

where ψ_γ is a vector state on $B(H_1^\gamma \otimes H_2^\gamma)$, H_1^γ, H_2^γ are finite dimensional and $u_i^\gamma : A(m,d) \to B(H_i^\gamma)$ are ucp maps for $i=1,2$.

By Lemma 3 there are K_1^γ, K_2^γ fin. dim and $*$-homomorphisms $\pi_i^\gamma : A(m,d) \to B(K_i^\gamma)$ s.t. $\forall x, y \in E(m,d)$ we have

$$u_1^\gamma(x) \otimes u_2^\gamma(y) = P_{H_1^\gamma\otimes H_2^\gamma}\,(\pi_1^\gamma(x)\otimes\pi_2^\gamma(y))_{|H_1^\gamma\otimes H_2^\gamma}$$

But this gives the desired conclusion since x_f is approximated by x_ξ for $\xi \in H_1^\gamma \otimes H_2^\gamma \subseteq K_1^\gamma \otimes K_2^\gamma$ and the PVMs obtained from π_1 and π_2. □

Proof of Proposition 3.

This proof was discussed last time and can be found in the notes of Lecture 8. ($\Rightarrow$) in a nutshell:

$A(m,d)$ has LLP since $C^*(\mathbb{Z}/m\mathbb{Z})$ has LLP and since LLP is preserved by free products.

$$C^*(\mathbb{F}_2) \otimes_{min} C^*(\mathbb{F}_2) \cong C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2) \implies C^*(\mathbb{F}_2) \text{ has WEP } + A(m,d) \text{ has LLP}$$

$$\implies C^*(\mathbb{F}_2) \otimes_{max} A(m,d) \cong C^*(\mathbb{F}_2) \otimes_{min} A(m,d)$$

$$\implies A(m,d) \text{ has WEP } + \text{ LLP}$$

$$\implies A(m,d) \otimes_{min} A(m,d) \cong A(m,d) \otimes_{max} A(m,d)$$

For $(\Leftarrow)$, note that when $d > 2$ or $m > 2$ then $\mathbb{F}_2 \subseteq \mathbb{Z}_m^{*d}$, hence we have, assuming $A(m,d) \otimes_{min} A(m,d) \cong A(m,d) \otimes_{max} A(m,d)$

$$\begin{tikzcd}
C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2) \arrow[r, phantom, "\cong"] & C^*(\mathbb{F}_2 \times \mathbb{F}_2) \arrow[r, two heads] \arrow[d, hook, "\text{injective}"] & C^*(\mathbb{F}_2) \otimes_{min} C^*(\mathbb{F}_2) \arrow[d, "\text{injective}"] \\
C^*(\mathbb{Z}_m^{*d}) \otimes_{max} C^*(\mathbb{Z}_m^{*d}) \arrow[r, phantom, "\cong"] & C^*(\mathbb{Z}_m^{*d} \times \mathbb{Z}_m^{*d}) \arrow[r, "\cong"] & C^*(\mathbb{Z}_m^{*d}) \otimes_{min} C^*(\mathbb{Z}_m^{*d})
\end{tikzcd}$$

Since bottom, left and right arrows are injective, so is the top one. $\square$

Proving that Tsirelson's Problem $\Rightarrow$ Kirchberg Conjecture requires some extra work. In particular, $\overline{Q_s(m,d)} = Q_c(m,d)$ implies that min and max norm coincide on the subspace $\{T + T^* \mid T \in E(m,d) \otimes_{alg} E(m,d)\}$. This relates to $C^*(\mathbb{F}_d) \otimes_{alg} C^*(\mathbb{F}_d)$, since it implies that min and max norm coincide on $\{T + T^* \mid T \in E(d) \otimes_{alg} E(d)\} \subseteq C^*(\mathbb{F}_d) \otimes_{alg} C^*(\mathbb{F}_d)$, where $E(d)$ is the linear span of the d free unitaries generating $C^*(\mathbb{F}_d)$. If this holds for every $d \in \mathbb{N}$, it holds for $d = \infty$ as well, and Ozawa showed that this entails that min and max norms are the same on $C^*(\mathbb{F}_\infty) \otimes_{alg} C^*(\mathbb{F}_\infty)$.