Nuclear C^*-algebras as inductive limits of finite dimensional C^*-algebras

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with Wilhelm Winter

WWU Münster

C^*-Algebras: Tensor Products, Approximation & Classification
In honour of Eberhard Kirchberg
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MM Mathematics Münster Cluster of Excellence

Westfälische Wilhelms-Universität Münster

CRC 1442 GEOMETRY: DEFORMATIONS AND RIGIDITY

Nuclear C^*-algebras

Theorem/Definition (Choi–Effros '78; Kirchberg '77)

A separable C^*-algebra A is nuclear iff there exists a sequence of finite-dimensional C^*-algebras $(F_n)_{n\in\mathbb{N}}$ and completely positive contractive (cpc) maps $\psi_n : A \to F_n$ and $\varphi_n : F_n \to A$ such that $\|\varphi_n \circ \psi_n(a) - a\| \to 0$ for all $a \in A$.

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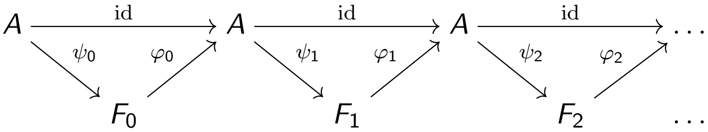

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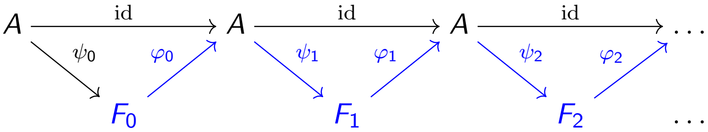

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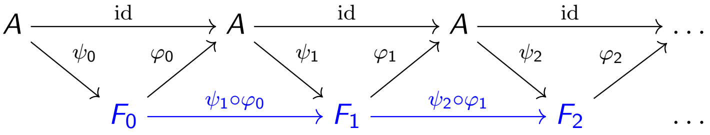

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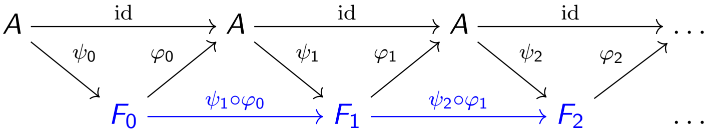

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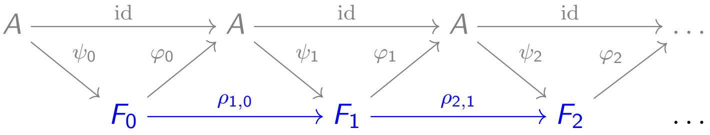

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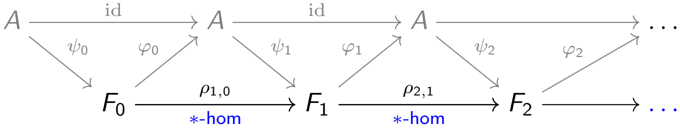
Forming the limit

Forming the limit with *-homomorphisms

Suppose the $\rho_{n+1,n}$ were *-homomorphisms.



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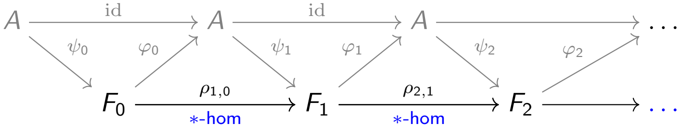

$$\frac{\prod F_j}{\bigoplus F_j}$$

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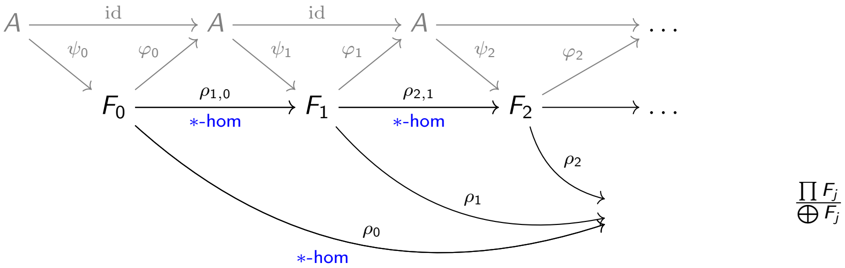

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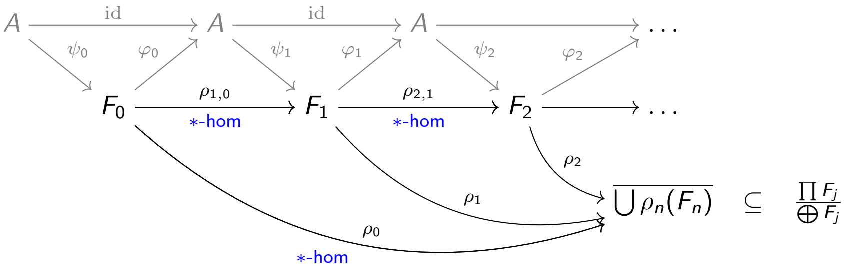

The limit of the system $(F_n, \rho_{n+1,n})_n$ is the C^*-subalgebra

$$\varinjlim(F_n, \rho_{n+1,n}) := \overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j}.$$

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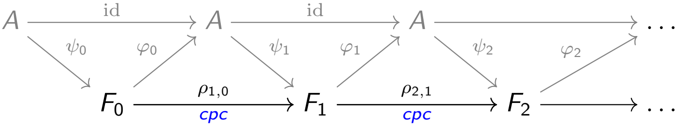

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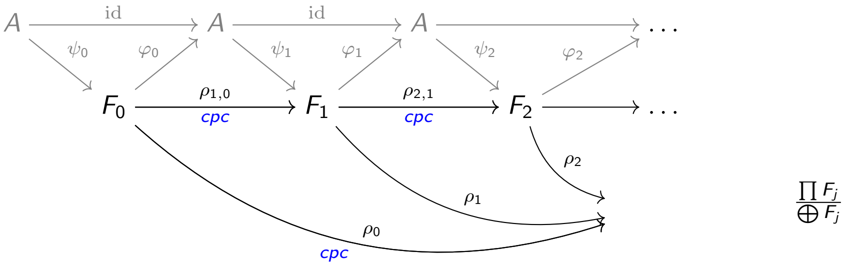

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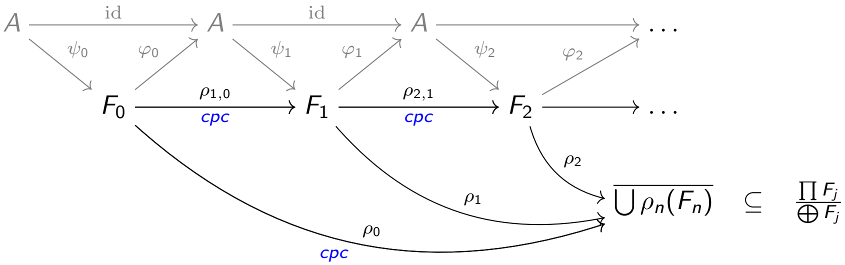

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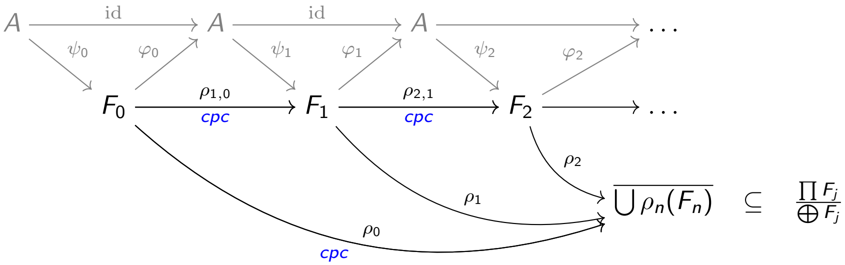

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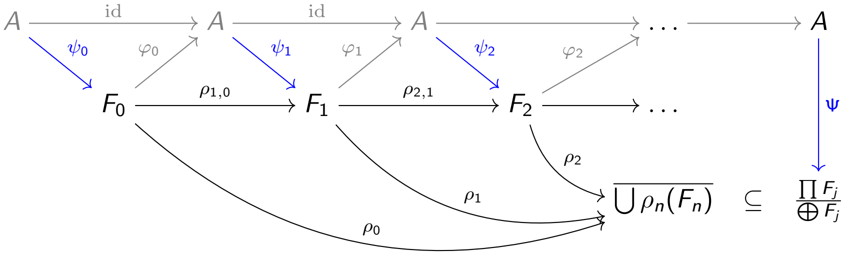
which is now just a closed self-adjoint subspace. How does it relate to A?

Back to A

The $(\psi_n)_n$ induce a cpc map $\Psi : A \to \prod F_j / \bigoplus F_j$.



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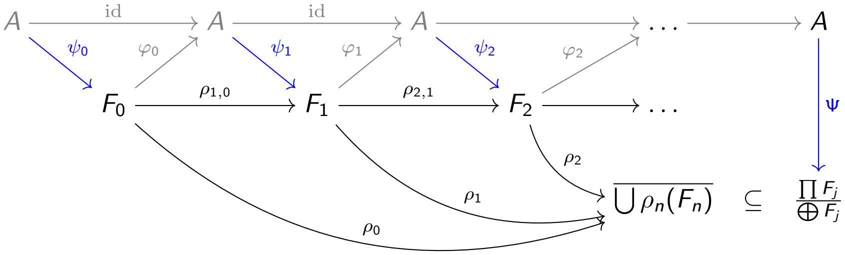

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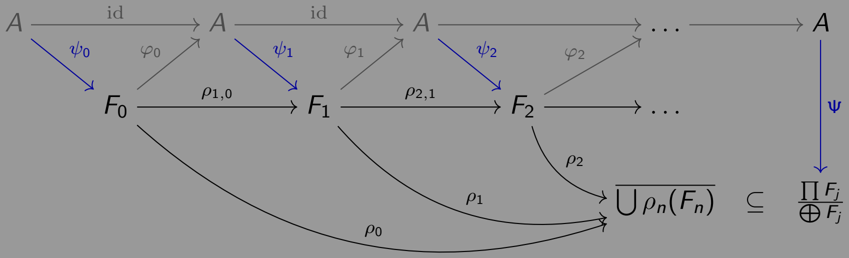

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$$\|\psi_n(a)\| \xrightarrow[n\to\infty]{} \|a\|, \ \forall\, a \in A$$

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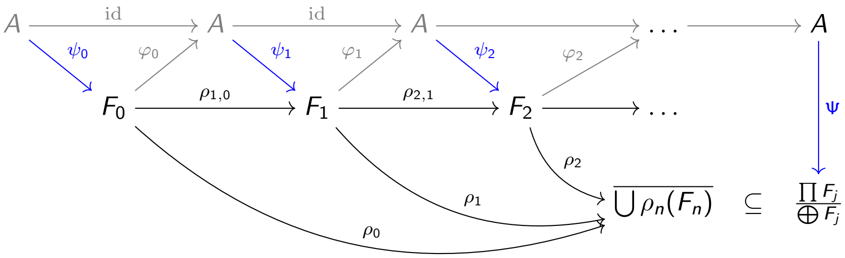

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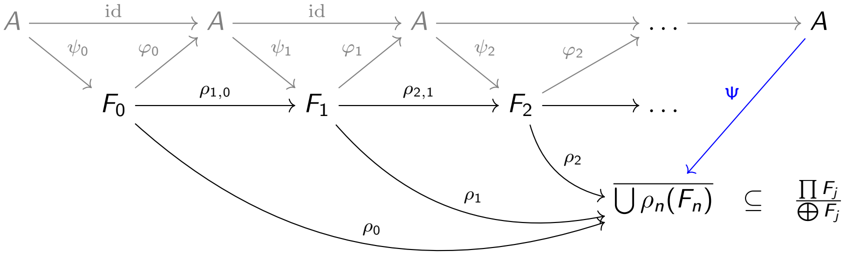

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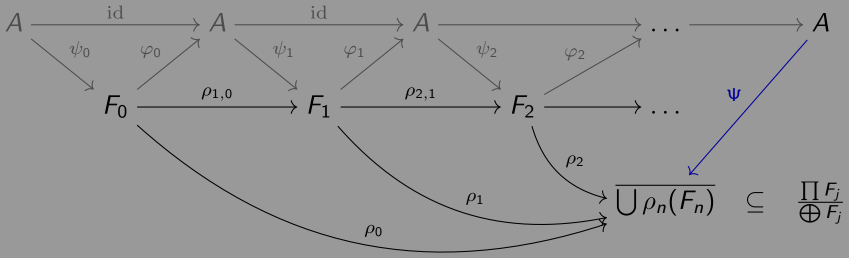

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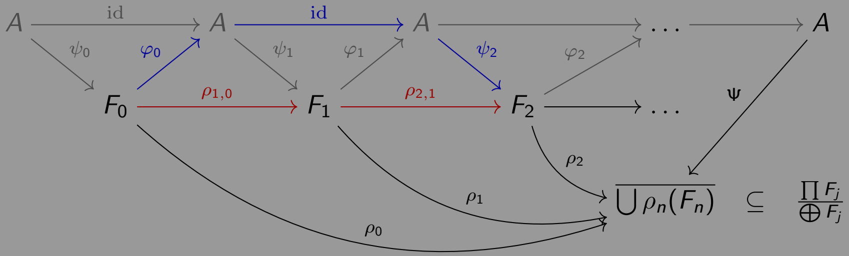

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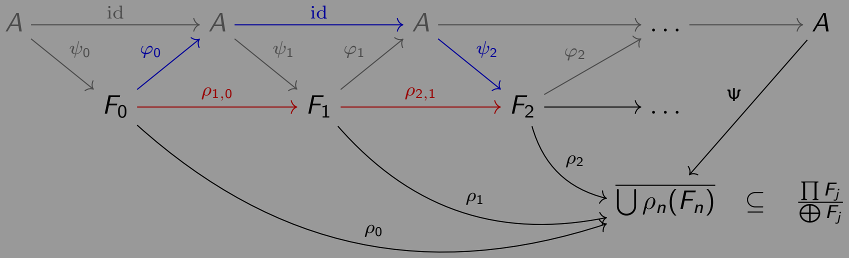

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Any system of cpc approximations admits a summable subsystem, so we assume our system is summable.

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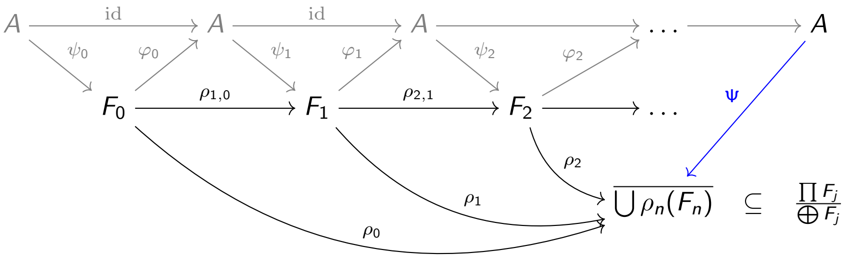

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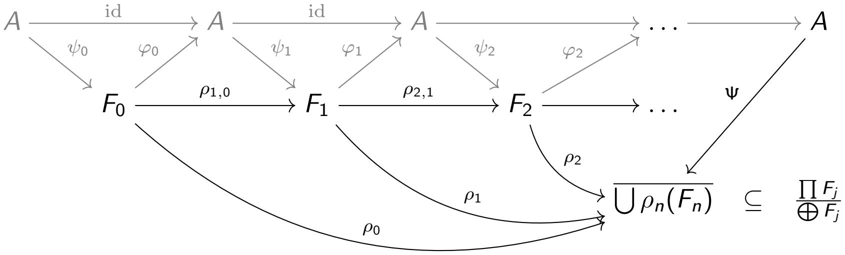

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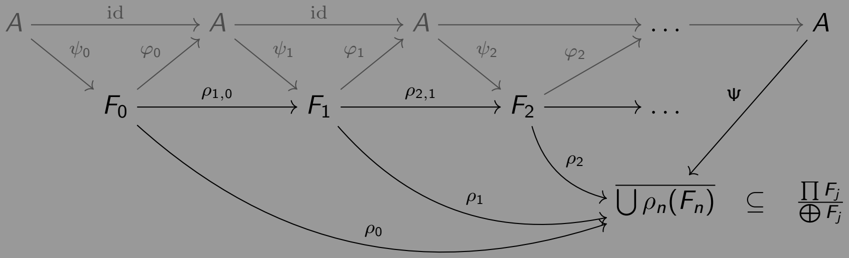

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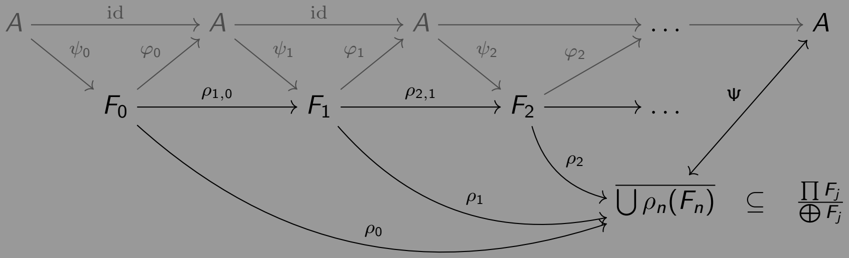

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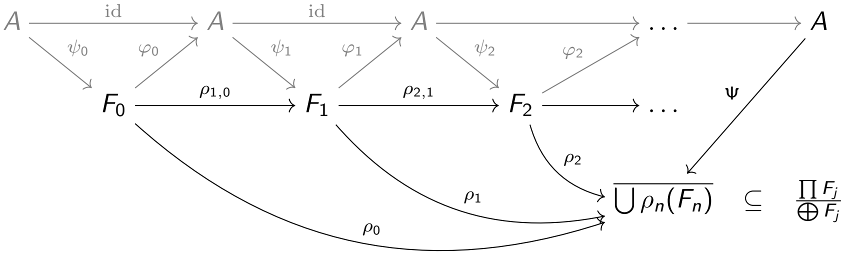

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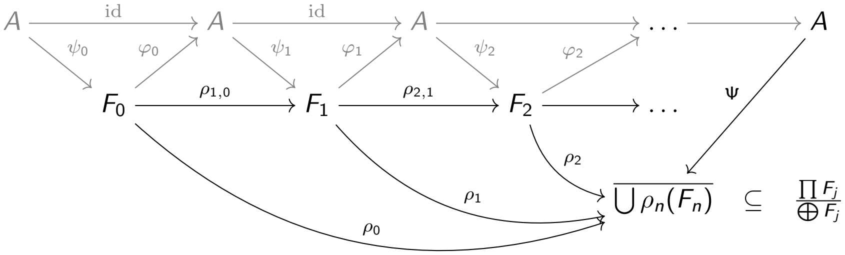

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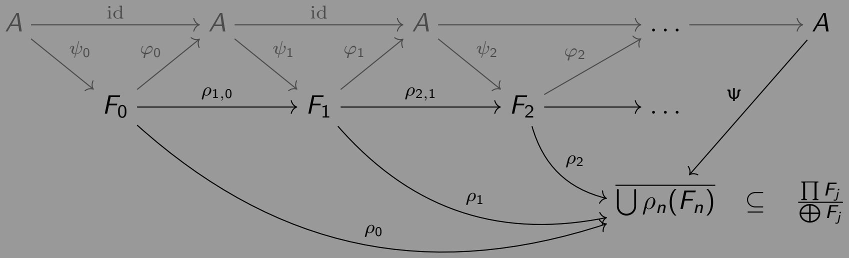

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That means Ψ is completely isometric and cp with cp inverse.

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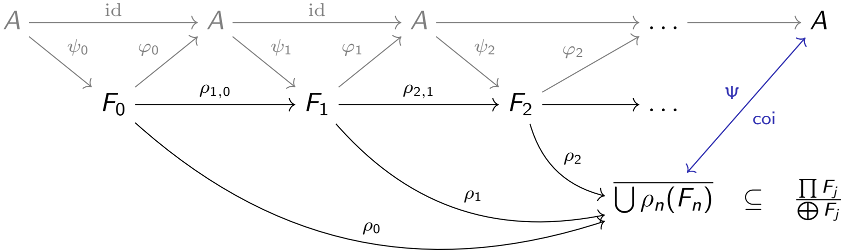

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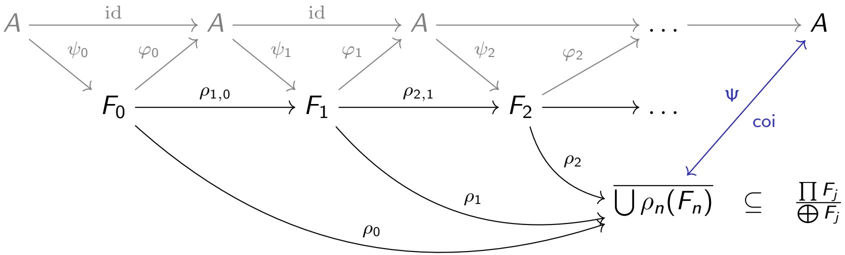

Since any coi between C^*-algebras is automatically a *-isomorphism, the coi class of a C^*-algebra captures its *-isomorphism class.

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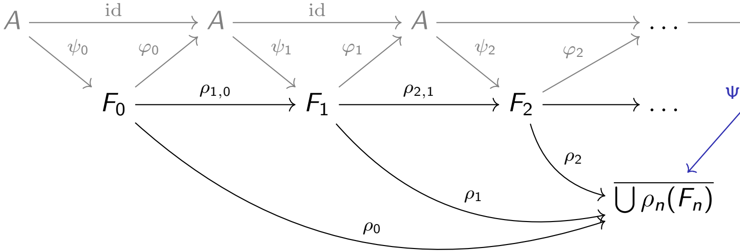

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Moreover, by equipping $\overline{\bigcup \rho_n(F_n)}$ with the product

$$\Psi(a) \bullet \Psi(b) := \Psi(ab), \ \forall \ a, b \in A,$$

we get a C^*-algebra $(\overline{\bigcup \rho_n(F_n)}, \bullet)$, which is *-isomorphic to A.

A nuclear C^*-algebra from a cpc system

Somehow this system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ produced, not a C^*-algebra, but a space completely order isomorphic to a nuclear C^*-algebra.

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In one sense this is a special case of Blackadar and Kirchberg's Generalized Inductive Systems. In another sense, it is a generalization.

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Question

Given a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, when is the limit $\overline{\bigcup \rho_n(F_n)}$ coi to a (nuclear) C^*-algebra?

Nuclearity

Proposition (C.–Winter, C.)

If the limit of a finite-dimensional cpc system is coi to a C^**-algebra* A*, then* A *is nuclear.*

Nuclearity

Proposition (C.–Winter, C.)

If the limit of a finite-dimensional cpc system is coi to a C^**-algebra* A*, then* A *is nuclear.*

This follows readily from Ozawa and Sato's One-Way-CPAP, which allows one to determine whether a given C^*-algebra A is nuclear by finding a certain family of cpc maps $\{\varphi_\lambda : F_\lambda \to A\}_\lambda$ from finite-dimensional C^*-algebras.

One Way CPAP

Theorem (Ozawa '02, Sato '21)

A C^*-algebra A is nuclear iff there exists a net $(\varphi_\lambda : F_\lambda \to A)_{\lambda\in\Lambda}$ of cpc maps from finite-dimensional C^*-algebras such that the induced cpc map

$$\begin{array}{ccc} \prod_\lambda F_\lambda & \xrightarrow{(\varphi_\lambda)_\lambda} & \ell^\infty(\Lambda, A) \\ \downarrow & & \downarrow \\ \prod_\lambda F_\lambda / \bigoplus_\lambda F_\lambda & \xrightarrow{\Phi} & \ell^\infty(\Lambda, A)/c_0(\Lambda, A) \end{array}$$

satisfies $A^1 \subset \Phi\left(\left(\frac{\prod_\lambda F_\lambda}{\bigoplus_\lambda F_\lambda}\right)^1\right)$.

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To get the φ_n in our case:

$$\begin{array}{ccccccc} F_0 & \xrightarrow{\rho_{1,0}} & F_1 & \xrightarrow{\rho_{2,1}} & F_2 & \longrightarrow & \dots \end{array}$$

$$\rho_0 : F_0 \to \overline{\bigcup \rho_n(F_n)}, \quad \rho_1 : F_1 \to \overline{\bigcup \rho_n(F_n)}, \quad \rho_2 : F_2 \to \overline{\bigcup \rho_n(F_n)}$$

$$\overline{\bigcup \rho_n(F_n)} \subseteq \frac{\prod F_j}{\bigoplus F_j}$$

$$\Psi : A \to \overline{\bigcup \rho_n(F_n)} \quad \text{coi}$$

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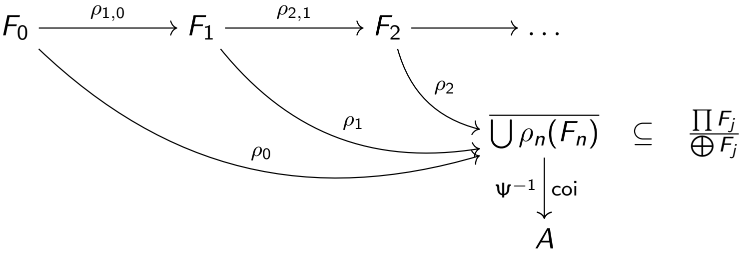

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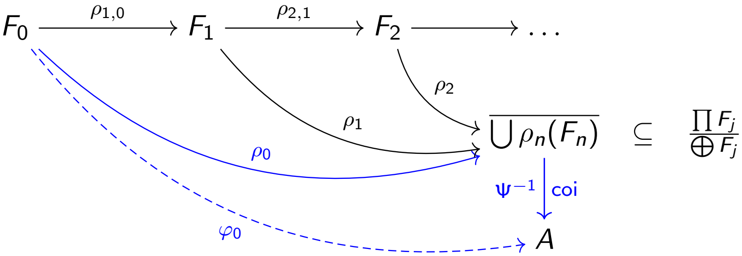

Back to our question

Question

Given a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, when is the limit $\overline{\bigcup \rho_n(F_n)}$ coi to a (nuclear) C^*-algebra?

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Think of this as saying that for $m > n > M$, the maps $\rho_{m,n}$ become more multiplicative on $\rho_{n,k}(x)$ and $\rho_{n,k}(y)$.

$$\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \|\rho_{m,n}\big(\rho_{n,k}(x)\big)\rho_{m,n}\big(\rho_{n,k}(y)\big)\| < \varepsilon.$$

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The limit $\overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j}$ is a C^*-subalgebra with multiplication

$$\rho_k(x)\rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)), \;\; k \geq 0, x, y \in F_k.$$

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The following are equivalent for a separable C^**-algebra* A*:*

1. A *is nuclear and quasidiagonal.*
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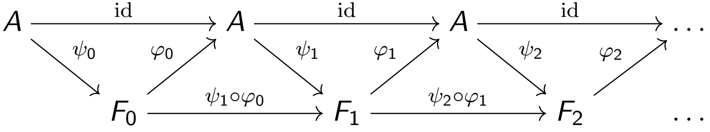
Moreover, for any nuclear and QD C^*-algebra A, there exists a system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ with $(\psi_n)_n$ approximately multiplicative so that the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is NF and its limit is *-isom to A.



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Proposition (Winter–Zacharias '09)

Let A and B be C^*-algebras with A unital. A cp map $\varphi : A \to B$ is order zero iff $\varphi(a)\varphi(b) = \varphi(1_A)\varphi(ab)$ for all $a, b \in A$.

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Note that a unital cp order zero map is automatically a *-homomorphism.

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CPC^*-systems (C.–Winter)

Definition (C.–Winter '23)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is CPC^* if it is asymptotically order zero, meaning that for any $k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, there exists an $M > k$ so that for all $m > n, j > M$

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NF and CPC^*-systems

Theorem (C.–Winter '23 (via Brown–Carrión–White))

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(Asymptotically/Approximately) multiplicative/ order zero maps carry significantly more structure than generic cpc maps. But these can be hard to get our hands on.

Though systems $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations with $(\psi_n)_n$ approximately multiplicative/ order zero are known to exist, they can be hard to find, and many well-known systems of cpc approximations do not produce NF or CPC^*-systems.

Systems from Følner sequences

For a countable, discrete, amenable group Γ, we can use any Følner sequence $(\mathcal{G}_n)_n$ to construct a system of ucp approximations of $\mathrm{C}^*_\lambda(\Gamma)$:

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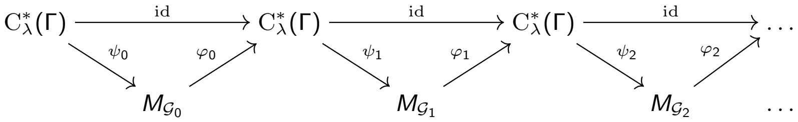

Identifying $M_{\mathcal{G}_n} \cong P_n B(\ell^2(\Gamma)) P_n$ with $P_n = \mathrm{proj}_{\mathrm{span}\{\delta_g | g \in \mathcal{G}_n\}}$, we set

$$\psi_n(x) = P_n x P_n \text{ for } x \in \mathrm{C}^*_\lambda(\Gamma) \subset B(\ell^2(\Gamma))$$

and

$$\varphi_n(e_{g,h}) = \frac{1}{|F_n|} \lambda_{gh^{-1}}$$

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If Γ has a non-torsion element (e.g. $\Gamma = \mathbb{Z}$), then the maps $(\psi_n)_n$ will be neither approximately multiplicative nor approximately order zero

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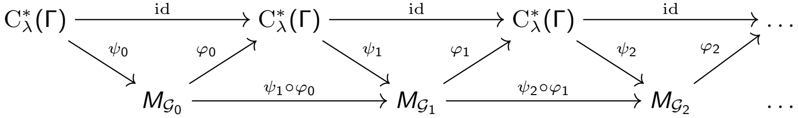

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If Γ has a non-torsion element (e.g. $\Gamma = \mathbb{Z}$), then the maps $(\psi_n)_n$ will be neither approximately multiplicative nor approximately order zero, and the resulting cpc system $(M_{\mathcal{G}_n}, \psi_{n+1} \circ \varphi_n)_n$ will neither be NF nor CPC^*.

Back to our motivating observations

(Asymptotically/Approximately) multiplicative/ order zero maps carry significantly more structure than generic cpc maps. But these can be hard to get our hands on.

Though systems $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations with $(\psi_n)_n$ approximately multiplicative/ order zero are known to exist, they can be hard to find, and many well-known systems of cpc approximations do not produce NF or CPC^*-systems.

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Question

Given a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, when is the limit $\overline{\bigcup \rho_n(F_n)}$ coi to a (nuclear) C^*-algebra?

C^*-encoding systems (C.)

Definition (C.'23)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is C^*-encoding if for any $k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, there exists an $M > k$ so that for all $m > n, j > M$

$$\|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,j}\big(\rho_{j,k}(x)\rho_{j,k}(y)\big)\| < \varepsilon.$$

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The limit $\overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j}$ is completely order isomorphic to the C^*-algebra $(\overline{\bigcup \rho_n(F_n)}, \bullet)$ with multiplication

$$\rho_k(x) \bullet \rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)), \; k \geq 0, x, y \in F_k.$$

All together

Definition (Blackadar–Kirchberg '97)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is NF if
$\forall\ k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, $\exists\ M > k$ so that $\forall\ m > n, j > M$

$$\|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,k}(x)\rho_{m,k}(y)\| < \varepsilon.$$

Definition (C.–Winter '23)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is CPC^* if
$\forall\ k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, $\exists\ M > k$ so that $\forall\ m > n, j > M$

$$\|\rho_{m,j}(1_{F_j})\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,k}(x)\rho_{m,k}(y)\| < \varepsilon.$$

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Theorem (C. '23)

The following are equivalent for a separable C^*-algebra A:

1. *A is nuclear.*
2. *A is coi to the limit of a C^*-encoding system.*

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Moreover, for any nuclear C^*-algebra A and any[1] system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations of A the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is C^*-encoding and its limit is coi to A.

[1] after possibly passing to a summable subsystem– same for NF and CPC^*

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Moreover, for any nuclear C^*-algebra A, there exists a system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ with $(\psi_n)_n$ approximately order zero so that the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is CPC^* and its limit is coi to A.

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Any NF system is C^*-encoding and (C.–Winter) has a CPC^*-subsystem.

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Question

Given a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, when is the limit $\overline{\bigcup \rho_n(F_n)}$ coi to a (nuclear) C^*-algebra?

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Given a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, when is the limit $\overline{\bigcup \rho_n(F_n)}$ coi to a (nuclear) C^*-algebra?

Theorem (C. '23)

For a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, the following are equivalent

1. *The limit is coi to a C^*-algebra.*
2. *The limit is coi to a nuclear C^*-algebra. (CW, OS)*
3. *The system has a C^*-encoding subsystem.*

Thank you.

Explicit example: $C^*_\lambda(\mathbb{Z})$

For $G = \mathbb{Z}$ and Følner sets $(\{0, \ldots, n-1\})_n$, we have $M_{\mathcal{G}_n} = M_n$ with matrix units $\{e_{i,j}\}_{i,j=0}^{n-1}$. Then for each n

$$\psi_n\left(\sum_{k\in\mathbb{Z}} a_k\lambda_k\right) = \psi_n\left(\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_{-1} & a_{-2} & \ddots \\ \ddots & a_1 & a_0 & a_{-1} & \ddots \\ \ddots & a_2 & a_1 & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}\right) = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & \cdots & \cdots & a_0 \end{bmatrix}.$$

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and $\varphi_n(e_{i,j}) = \frac{1}{n}\lambda_{i-j}$. back
For $m > n \geq 0$ the compositions $\rho_{m,n}$ are given on matrix units by

$$\rho_{m,n}(e_{i,j}) = \frac{1}{n}\left(\prod_{k=1}^{m-1} 1 - \frac{|i-j|}{n+k}\right) S_m^{i-j},$$

where $S_n \in M_n$ is the shift.

Summability

A system of c.p.c. approximations $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of a separable C^*-algebra A is summable if there exists a decreasing sequence $(\varepsilon_n) \in \ell^1(\mathbb{N})^1_+$ so that $\|\varphi_n - \varphi_m \circ \psi_m \circ \varphi_n\| < \epsilon_n$ for all $m > n \geq 0$.

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We will call a Følner sequence $(\mathcal{G}_n)_n$ for a discrete group G summable if there exists a decreasing sequence $(\varepsilon_n) \in \ell^1(\mathbb{N})^1_+$ so that for all $m > n \geq 0$

$$\max_{g,h \in \mathcal{G}_n} \left(1 - \frac{|\mathcal{G}_m \cap gh^{-1}\mathcal{G}_m|}{|\mathcal{G}_m|}\right) |\mathcal{G}_n| < \varepsilon_m.$$

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One sub-Følner sequence of $(\{0, ..., n\})_n$ for $\mathbb{Z}$ making the system of cpc approximations from before summable (for $\varepsilon_n = 2^{n+1}$) is given by $\mathcal{G}_0 = \{0\}$ and $\mathcal{G}_n = \{0, ..., 2^n|\mathcal{G}_{n-1}|\}$ for $n \geq 1$.

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$$\varphi_n(\psi_n(\lambda_k)) = \varphi_n(S^k_{|\mathcal{G}_n|}) = \frac{|\mathcal{G}_n| - |k|}{|\mathcal{G}_n|}\lambda_k$$

for $n > k \geq 0$ where $S_{|\mathcal{G}_n|} \in M_{|\mathcal{G}_n|}$ is the shift. A few iterations yields

$$\rho_{m,n}(e_{i,j}) = \frac{1}{|\mathcal{G}_n|}\left(\prod_{k=1}^{m-1} \frac{|\mathcal{G}_{n+k}| - |i-j|}{|\mathcal{G}_{n+k}|}\right) S^{i-j}_{|\mathcal{G}_m|}, \quad \text{for } m > n \geq 0, 0 \leq i, j \leq n.$$