

Nuclear C^* -algebras and generalized inductive limits

Kristin Courtney

based in part on joint work with W. Winter
and with N. Galke, L. van Luijk, and A. Stottmeister

UC Berkeley Probabilistic Operator Algebra Seminar

Part I: Inductive limits and AF algebras

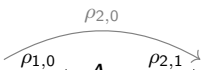
Inductive limits of C^* -algebras

An **inductive system** of C^* -algebras consists of a sequence $(A_n)_n$ of C^* -algebras together with connecting $*$ -homomorphisms

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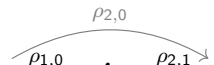
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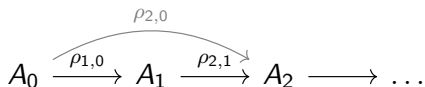
$$(\rho_{n,k}(a))_n \in \prod_n A_n.$$

The quotient map $\prod_n A_n \rightarrow \prod A_n / \bigoplus A_n$ induces $*$ -homomorphisms $\rho_k : A_k \rightarrow \prod A_n / \bigoplus A_n$ for each $k \geq 0$ by

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The **inductive limit** of the system $(A_n, \rho_{m,n})$ is the C^* -algebra

$$\varinjlim (A_n, \rho_{m,n}) := \overline{\bigcup_k \rho_k(A_k)} \subset \prod A_n / \bigoplus A_n.$$

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The CAR algebra:

$$M_2 \xrightarrow{a \mapsto a \oplus a} M_4 \hookrightarrow \dots \hookrightarrow \overline{\bigcup_k M_{2^k}} =: M_{2^\infty}$$

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Alternatively, an AF C^* -algebra is one that contains an ascending sequence of finite-dimensional subalgebras with norm-dense union, which makes its von Neumann analogue a little more apparent.

AFD von Neumann Algebras

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A von Neumann algebra is called **Approximately Finite Dimensional (AFD)** (or **hyperfinite**) if it contains an ascending sequence of finite-dimensional von Neumann subalgebras with weak*-dense union.

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Example

The hyperfinite II_1 -factor $\mathcal{R}$.

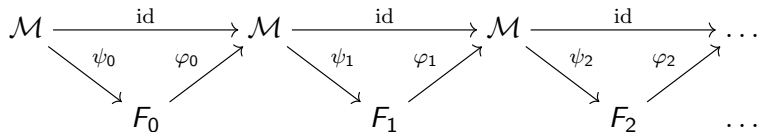
$$M_2 \xrightarrow{a \mapsto a \oplus a} M_4 \hookrightarrow \dots \hookrightarrow \overline{\bigcup_k M_{2^k}}^{wk^*} = \mathcal{R}$$

Semi-discrete von Neumann Algebras

Definition

A separably acting von Neumann algebra $\mathcal{M}$ is **semi-discrete** iff there exists a sequence of finite-dimensional von Neumann algebras $(F_n)_{n \in \mathbb{N}}$ and unital completely positive (ucp) maps $\mathcal{M} \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} \mathcal{M}$ such that $\varphi_n \circ \psi_n \rightarrow \text{id}_{\mathcal{M}}$ pointwise wk*.

This gives a sequence of (wk*-)approximately commuting diagrams.

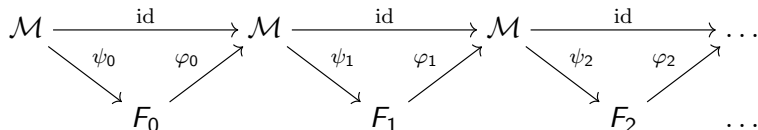


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Example

Any AFD von Neumann algebra is semi-discrete.

Nuclear C^* -algebras

Theorem/Definition (Choi–Effros '78; Kirchberg '77)

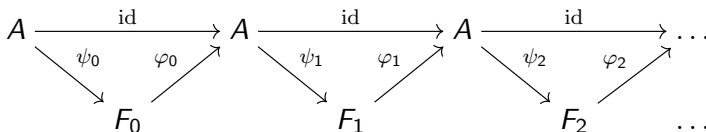
A separable C^* -algebra A is **nuclear** iff there exists a sequence of finite-dimensional C^* -algebras $(F_n)_{n \in \mathbb{N}}$ and completely positive contractive (cpc) maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ such that $\varphi_n \circ \psi_n \rightarrow \text{id}_A$ pointwise in norm.

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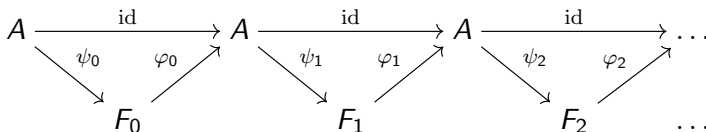
We call $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ a **system of cpc approximations** of A .

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Example

Any AF C^* -algebra is nuclear.

Connes' Theorem

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Example

- $C(X)$ where $\dim(X) \geq 1$.
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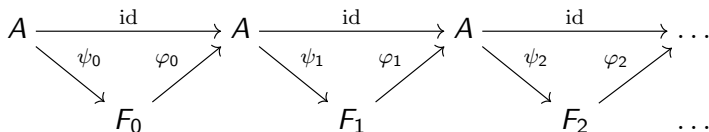
Even though a direct analogue to Connes' result, i.e., “nuclear $\Rightarrow$ AF”, is out of the question, any system of cpc approximations of a nuclear C^* -algebra gives rise to something very much like an inductive system.

From CPAP to an “inductive system”

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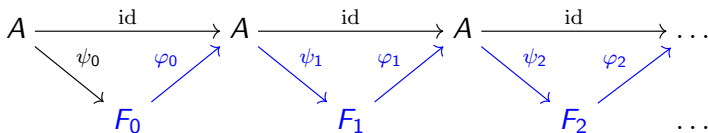


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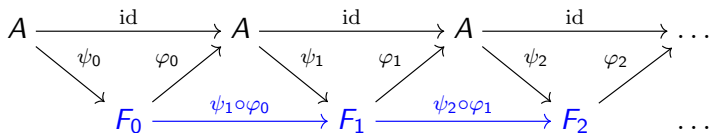


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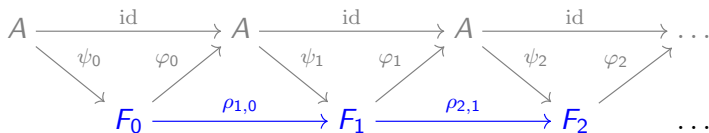


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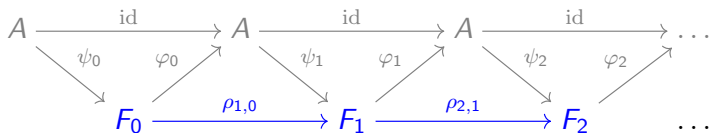
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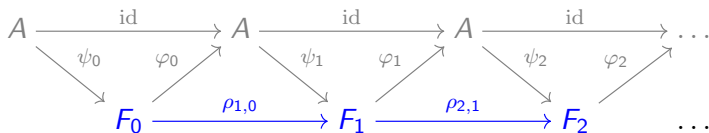
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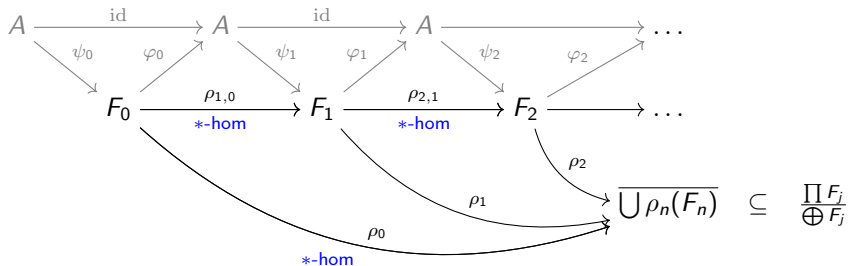
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Forming the limit

Forming the limit with $*$ -homomorphisms

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Suppose $A = \overline{\bigcup F_n}$ with $\varphi_n : F_n \hookrightarrow A$ the inclusion and $\psi_n : A \rightarrow F_n$ a conditional expectation. Then the $\rho_{n+1,n}$ are $*$ -homomorphisms,

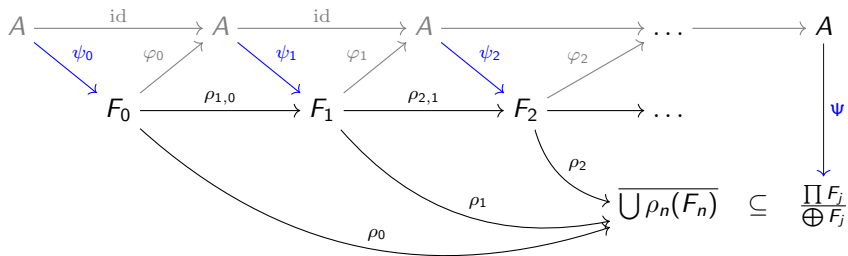


and $(F_n, \rho_{n+1,n})_n$ is an inductive system with limit

$$\varinjlim (F_n, \rho_{n+1,n}) := \overline{\bigcup \rho_n(F_n)} \subset \prod_{\oplus F_j}.$$

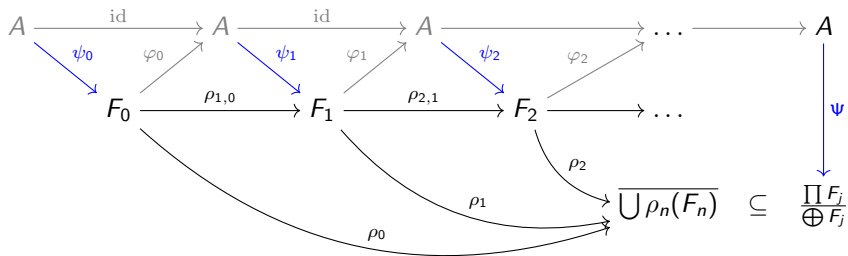
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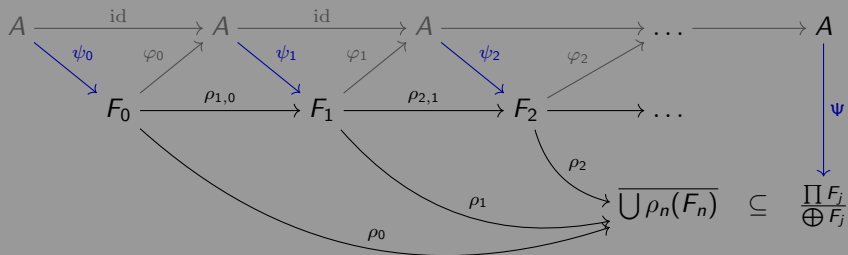


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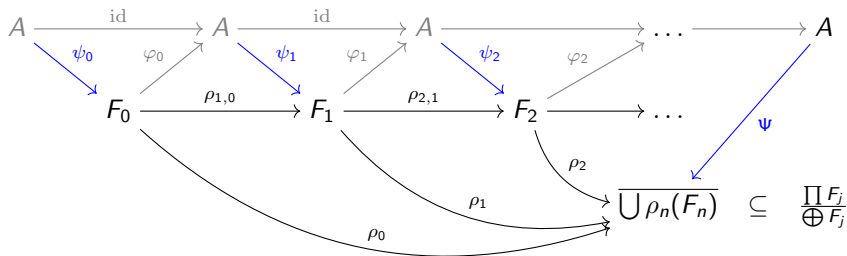
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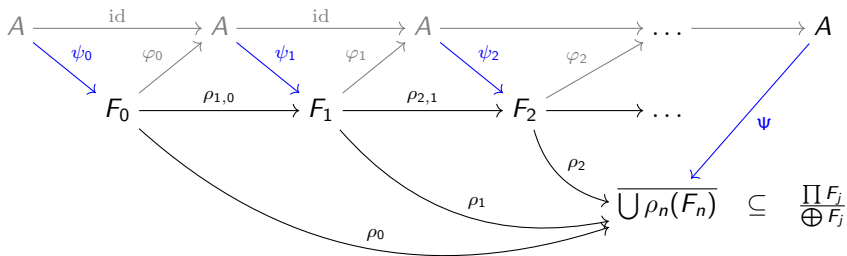
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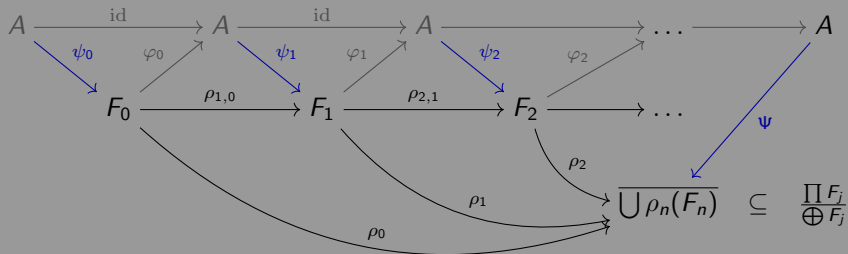
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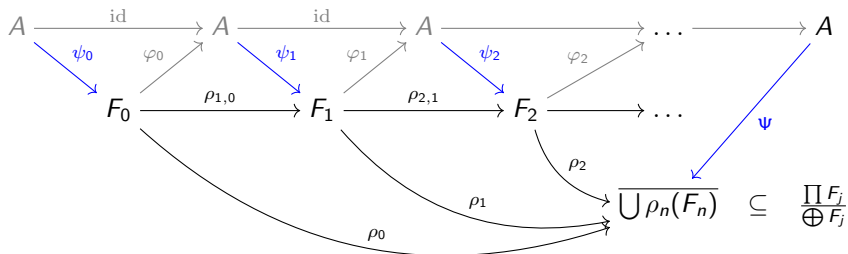
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$$\|\psi_n(ab) - \psi_n(a)\psi_n(b)\| \xrightarrow{n \rightarrow \infty} 0, \forall a, b \in A$$

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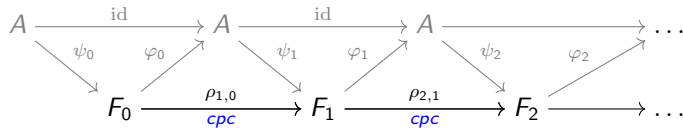
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$\rightsquigarrow \Psi : A \rightarrow \overline{\bigcup \rho_n(F_n)}$ is a $*$ -isomorphism.

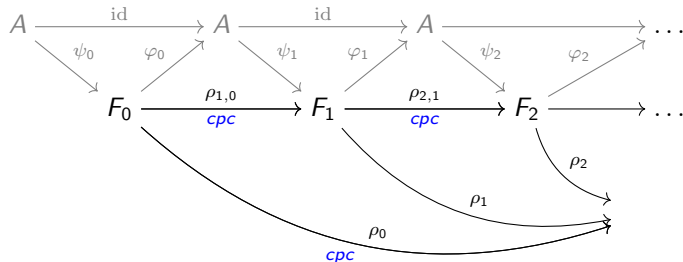
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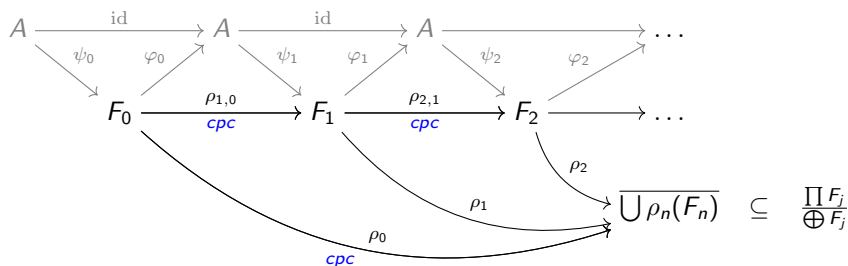
When the $\rho_{n+1,n}$ are **cpc** maps, they still induce **cpc** maps $\rho_n : F_n \rightarrow \prod F_j / \bigoplus F_j$ with $\rho_n(x) = [(\rho_{m,n}(x))_{m>n}]$.



$$\frac{\prod F_j}{\bigoplus F_j}$$

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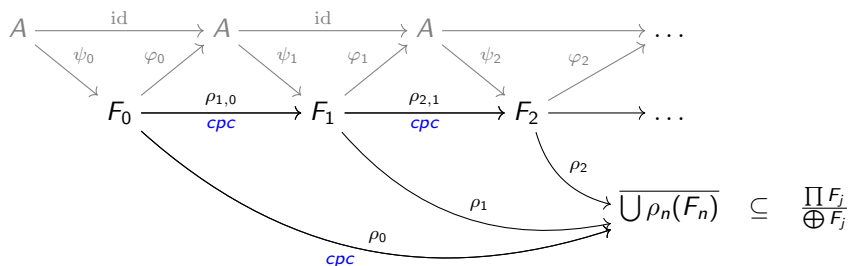
We can still form the limit of the system $(F_n, \rho_{n+1,n})_n$

$$\overline{\bigcup \rho_n(F_n)} \subset \frac{\Pi F_j}{\bigoplus F_j},$$

which is now just a closed self-adjoint subspace.

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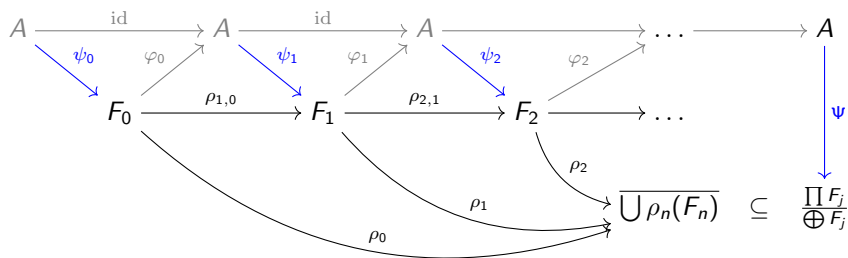
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How does it relate to A ?

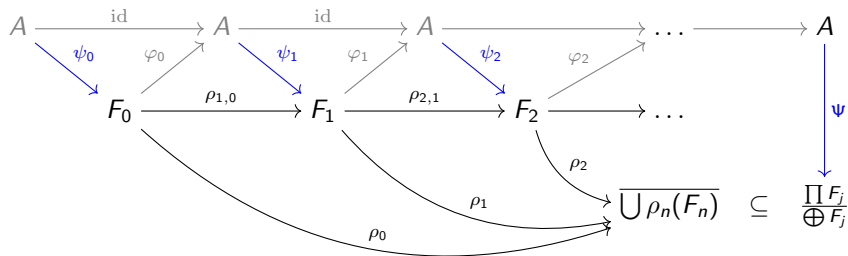
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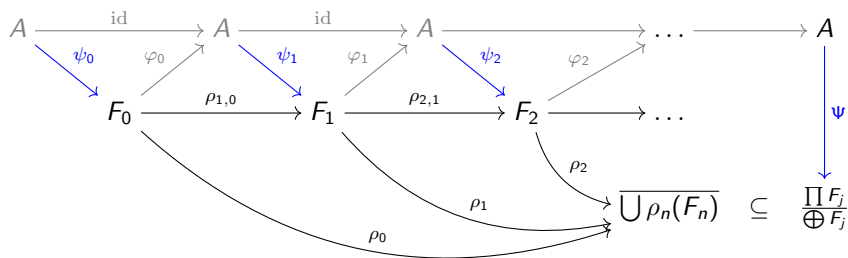


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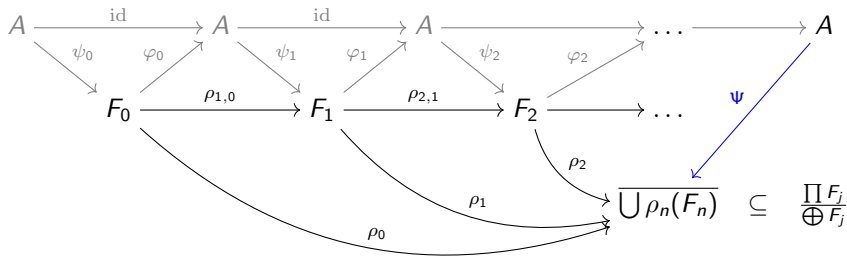
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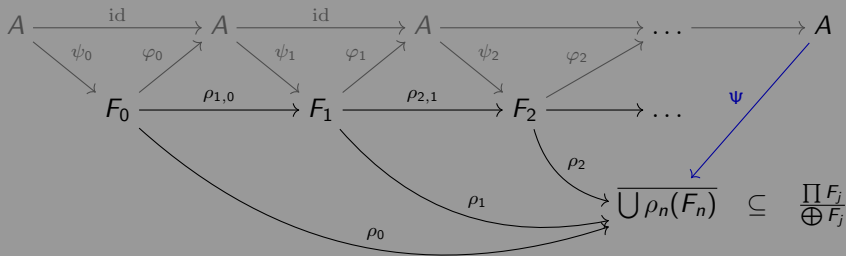


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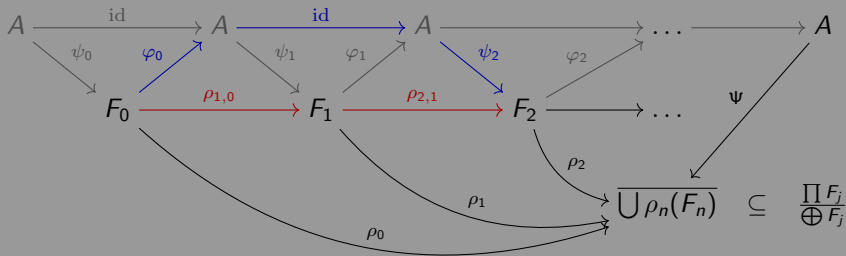
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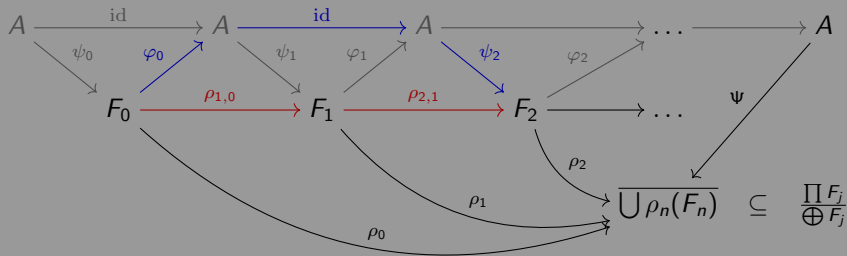
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$\Psi(A) = \overline{\bigcup \rho_n(F_n)}$ if $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ is summable

$$\|\rho_{m,n} - \psi_m \circ \varphi_n\| \xrightarrow{m,n \rightarrow \infty} 0$$

Back to A

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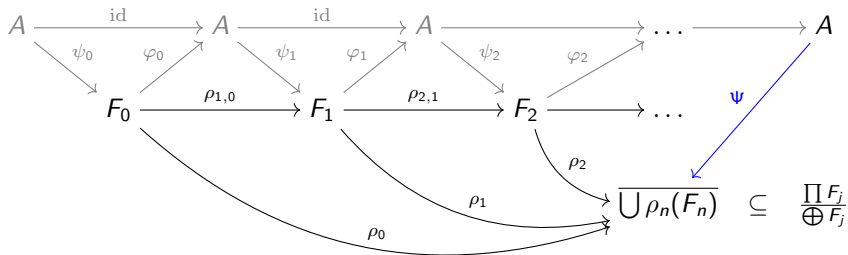
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Any system of cpc approximations admits a summable subsystem, so we assume our system is summable.

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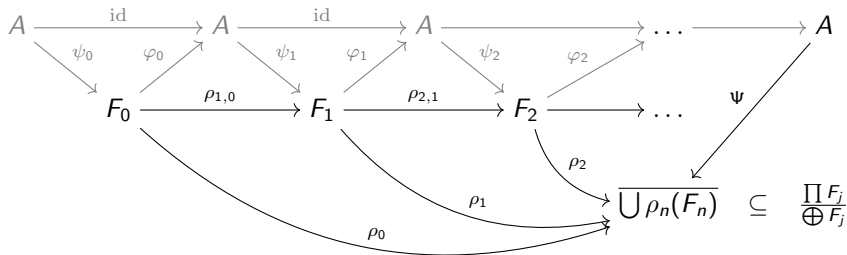


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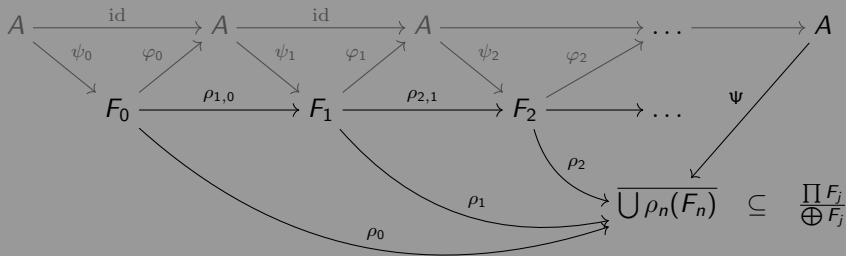
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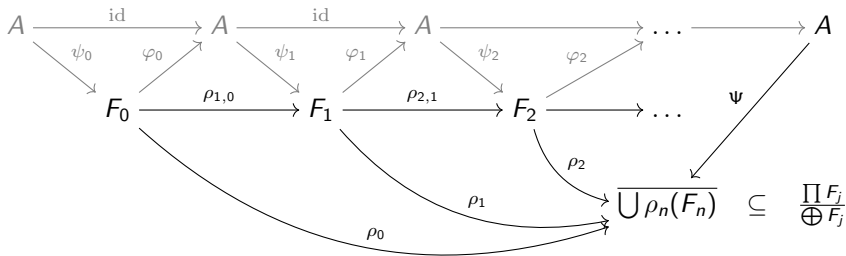
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This can only happen if A is quasidiagonal (QD).

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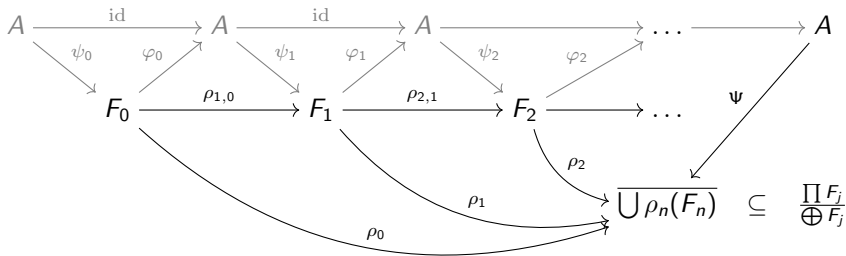


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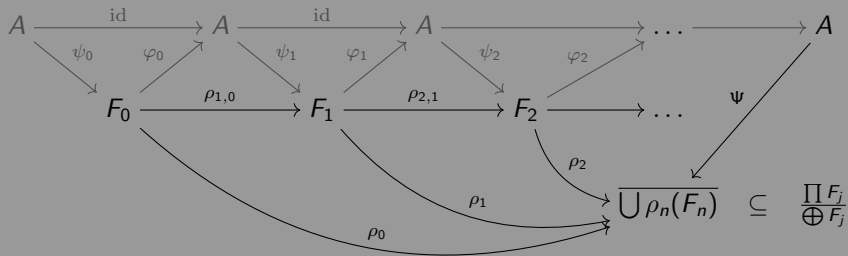
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$\rightsquigarrow \Psi : A \rightarrow \overline{\bigcup \rho_n(F_n)}$ is a complete order isomorphism (coi).

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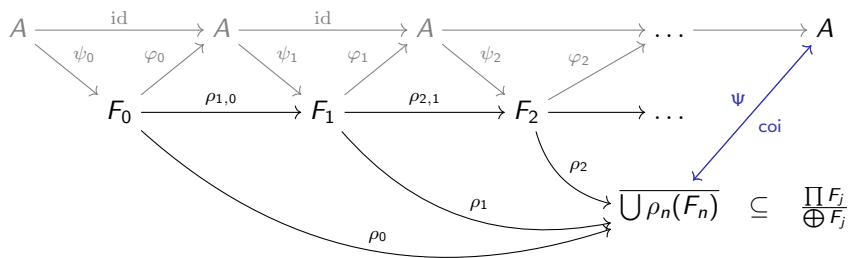
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That means Ψ is completely isometric and cp with cp inverse.

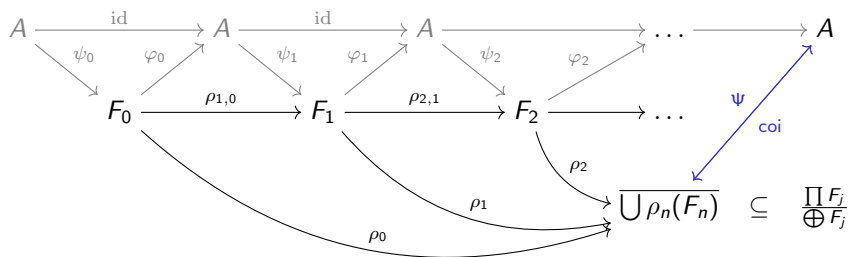
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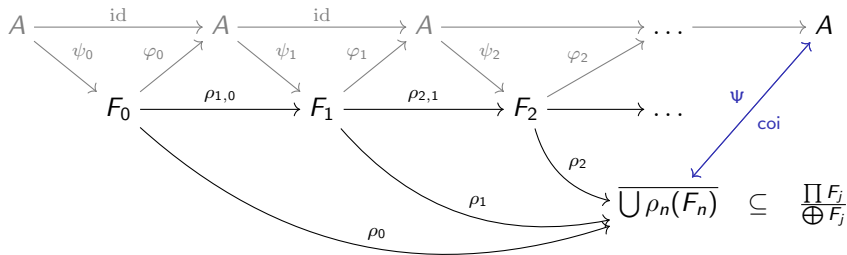
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Since any coi between C^* -algebras is automatically a $*$ -isomorphism, the coi class of a C^* -algebra captures its $*$ -isomorphism class.

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Since any coi between C^* -algebras is automatically a $*$ -isomorphism, the coi class of a C^* -algebra captures its $*$ -isomorphism class.

Moreover, by equipping $\overline{\bigcup \rho_n(F_n)}$ with the product

$$\Psi(a) \bullet \Psi(b) := \Psi(ab), \quad \forall a, b \in A,$$

we get a C^* -algebra $(\overline{\bigcup \rho_n(F_n)}, \bullet)$, which is $*$ -isomorphic to A .

Part II: cpc systems and nuclearity

cpc systems

Somehow the system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ produced, not a C^* -algebra, but a space completely order isomorphic to a nuclear C^* -algebra.

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Definition

We call a sequence of C^* -algebras $(A_n)_n$ together with cpc connecting maps $\rho_{n+1,n} : A_n \rightarrow A_{n+1}$ a **cpc system**, denoted $(A_n, \rho_{n+1,n})_n$. When the A_n are all finite-dimensional, we call the system **finite-dimensional**.

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In one sense this is a special case of Blackadar and Kirchberg's Generalized Inductive Systems. In another sense, it is a generalization.

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Question

Given a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, when is the limit $\overline{\bigcup \rho_n(F_n)}$ coi to a (nuclear) C^* -algebra?

Nuclearity

Proposition (C.–Winter, C.)

If the limit of a finite-dimensional cpc system is coi to a C^ -algebra A , then A is nuclear.*

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This follows readily from Ozawa and Sato's One-Way-CPAP, which allows one to determine whether a given C^* -algebra A is nuclear by finding a certain family of cpc maps $\{\varphi_\lambda : F_\lambda \rightarrow A\}_\lambda$ from finite-dimensional C^* -algebras.

One Way CPAP

Theorem (Ozawa '02, Sato '21)

A C^ -algebra A is nuclear iff there exists a net $(\varphi_\lambda : F_\lambda \rightarrow A)_{\lambda \in \Lambda}$ of cpc maps from finite-dimensional C^* -algebras such that the induced cpc map*

$$\begin{array}{ccc}
 \prod_\lambda F_\lambda & \xrightarrow{(\varphi_\lambda)_\lambda} & \ell^\infty(\Lambda, A) \\
 \downarrow \Downarrow & & \downarrow \Downarrow \\
 \prod_\lambda F_\lambda / \bigoplus_\lambda F_\lambda & \xrightarrow{\Phi} & \ell^\infty(\Lambda, A) / c_0(\Lambda, A)
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 \quad \text{satisfies } A^1 \subset \Phi \left(\left(\frac{\prod_\lambda F_\lambda}{\bigoplus_\lambda F_\lambda} \right)^1 \right).$$

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To get the φ_n in our case:

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Think of this as saying that for $m > n > M$, the maps $\rho_{m,n}$ become more multiplicative on $\rho_{n,k}(x)$ and $\rho_{n,k}(y)$.

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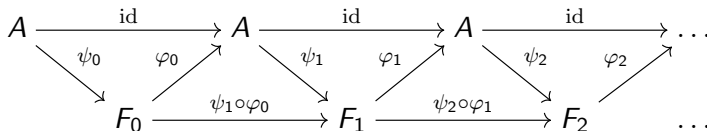
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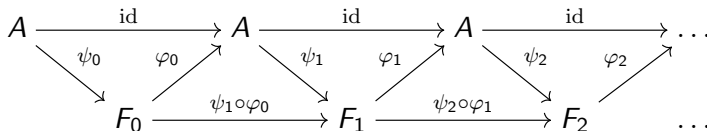
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Remark

For any summable system of cpc approximations $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$, $(\psi_n)_n$ are approximately multiplicative iff $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is NF.

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Note that a unital cp order zero map is automatically a $*$ -homomorphism.

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The limit $\overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j}$ of an **NF system** is a **C*-subalgebra** with

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The limit $\overline{\bigcup \rho_n(F_n)} \subset \overline{\prod_{\oplus F_j}}^{\oplus F_j}$ of a **CPC*** system is **completely order isomorphic to the C*-algebra $(\overline{\bigcup \rho_n(F_n)}, \bullet)$** with

$$\rho_k(x) \bullet \rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)), \quad \forall k \geq 0, x, y \in F_k.$$

Side-by-side

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NF and CPC^* -systems

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*Moreover, for any **nuclear and QD** C^* -algebra A , there exists a system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ with $(\psi_n)_n$ approximately **multiplicative** so that the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is **NF** and its limit is **$*$ -isom** to A .*

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Theorem (C.–Winter '23 (via Blackadar-Kirchberg + Voiculescu)

The following are equivalent for a separable C^* -algebra A :

1. A *nuclear*.
2. A is *coi* to the limit of a *CPC^* -system*.

Moreover, for any *nuclear* C^* -algebra A , there exists a system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ with $(\psi_n)_n$ approximately *order zero* so that the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is *CPC^** and its limit is *coi* to A .

Hierarchy

Example (C.–Winter)

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Any separable nuclear non-QD C^* -algebra admits a system

$(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ so that the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is CPC^* and not NF.

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Remark

If the connecting maps $\rho_{n+1,n} : F_n \rightarrow F_{n+1}$ are unital, then a CPC^ system is automatically NF.*

Part III: C^* -encoding systems

Back to our motivating observations

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NF and CPC^* -systems

Theorem (Blackadar–Kirchberg '97)

The following are equivalent for a separable C^ -algebra A :*

- 1. A is nuclear and QD.*
- 2. A is $*$ -isomorphic to the limit of an NF system.*

Moreover, for any nuclear and QD C^ -algebra A , there exists a system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ with $(\psi_n)_n$ approximately multiplicative so that the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is NF and its limit is $*$ -isom to A .*

Theorem (C.–Winter '23 (via Blackadar-Kirchberg + Voiculescu))

The following are equivalent for a separable C^ -algebra A :*

- 1. A nuclear.*
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Though systems $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations with $(\psi_n)_n$ approximately multiplicative/ order zero are known to exist, they can be hard to find, and many well-known systems of cpc approximations do not produce NF or CPC*-systems.

Systems from Følner sequences

For a countable, discrete, amenable group Γ , we can use any Følner sequence $(\mathcal{G}_n)_n$ to construct a system of ucp approximations of $C_\lambda^*(\Gamma)$:

$$C_\lambda^*(\Gamma) \xrightarrow{\text{id}} C_\lambda^*(\Gamma) \xrightarrow{\text{id}} C_\lambda^*(\Gamma) \xrightarrow{\text{id}} \dots$$

 $\searrow \psi_0 \quad \nearrow \varphi_0 \qquad \searrow \psi_1 \quad \nearrow \varphi_1 \qquad \searrow \psi_2 \quad \nearrow \varphi_2 \qquad \dots$
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Identifying $M_{\mathcal{G}_n} \cong P_n B(\ell^2(\Gamma)) P_n$ with $P_n = \text{proj}_{\text{span}\{\delta_g \mid g \in \mathcal{G}_n\}}$, we set

$$\psi_n(x) = P_n x P_n \text{ for } x \in C_\lambda^*(\Gamma) \subset B(\ell^2(\Gamma))$$

and

$$\varphi_n(e_{g,h}) = \frac{1}{|\mathcal{F}_n|} \lambda_{gh^{-1}}$$

where $\{e_{g,h} \mid g, h \in \mathcal{G}_n\} \subset M_{\mathcal{G}_n}$ are the matrix units.

Example $C_\lambda^*(\mathbb{Z})$

For $\Gamma = \mathbb{Z}$ and Følner sets $\mathcal{G}_n = \{0, \dots, n-1\}$, we have $M_{\mathcal{G}_n} = M_n$ with matrix units $\{e_{i,j}\}_{i,j=0}^{n-1}$. Then for each n

$$\psi_n \left(\sum_{k \in \mathbb{Z}} a_k \lambda_k \right) = \psi_n \left(\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_{-1} & a_{-2} & \ddots \\ \ddots & a_1 & a_0 & a_{-1} & \ddots \\ \ddots & a_2 & a_1 & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \right) = \begin{bmatrix} a_0 & a_{-1} & \dots & a_{-(n-1)} \\ a_1 & a_0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & \dots & \dots & a_0 \end{bmatrix}.$$

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where $\{e_{g,h} \mid g, h \in \mathcal{G}_n\} \subset M_{\mathcal{G}_n}$ are the matrix units.

Proposition (C.)

If Γ has a non-torsion element (e.g. $\Gamma = \mathbb{Z}$), then the maps $(\psi_n)_n$ will not be approximately multiplicative/ order zero, and the resulting cpc system $(M_{\mathcal{G}_n}, \psi_{n+1} \circ \varphi_n)_n$ will neither be NF nor CPC^* .

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and $\varphi_n(e_{i,j}) = \frac{1}{n} \lambda_{i-j}$. For the bilateral shift λ_1 ,

$$\|\psi_n(\lambda_1^* \lambda_1) - \psi_n(\lambda_1)^* \psi_n(\lambda_1)\| = \|e_{n,n}\| = 1.$$

Back to our motivating observations

(Asymptotically/Approximately) multiplicative/ order zero maps carry significantly more structure than generic cpc maps. But these can be hard to get our hands on.

Though systems $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations with $(\psi_n)_n$ approximately multiplicative/ order zero are known to exist, they can be hard to find, and many well-known systems of cpc approximations do not produce NF or CPC*-systems.

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However, we saw that any system $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations produces (after possibly passing to a summable subsystem) a cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ whose limit is completely order isomorphic to a nuclear C^* -algebra.

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Question

Given a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, when is the limit $\overline{\bigcup \rho_n(F_n)}$ coi to a (nuclear) C^* -algebra?

C*-encoding systems (C.)

Definition (C.'23)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is **C*-encoding** if for any $k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, there exists an $M > k$ so that for all $m > n, j > M$

$$\|\rho_{m,n}(\rho_{n,k}(x)\rho_{n,k}(y)) - \rho_{m,j}(\rho_{j,k}(x)\rho_{j,k}(y))\| < \varepsilon.$$

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The limit $\overline{\bigcup \rho_n(F_n)} \subset \frac{\prod F_j}{\bigoplus F_j}$ is completely order isomorphic to the C^* -algebra $(\overline{\bigcup \rho_n(F_n)}, \bullet)$ with

$$\rho_k(x) \bullet \rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)), \quad \forall k \geq 0, x, y \in F_k.$$

All together

Definition (Blackadar–Kirchberg '97)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is **NF** if

$\forall k \geq 0, x, y \in F_k$, and $\varepsilon > 0$, $\exists M > k$ so that $\forall m > n, j > M$

$$\|\rho_{m,n}(\rho_{n,k}(x)\rho_{n,k}(y)) - \rho_{m,k}(x)\rho_{m,k}(y)\| < \varepsilon.$$

Definition (C.–Winter '23)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is **CPC*** if

$\forall k \geq 0, x, y \in F_k$, and $\varepsilon > 0$, $\exists M > k$ so that $\forall m > n, j > M$

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Example

From our Følner approximation of $C_\lambda^*(\mathbb{Z})$, the compositions $\rho_{m,n}$ are given on matrix units by (with $S_n \in M_n$ is the shift)

$$\rho_{m,n}(e_{i,j}) = \frac{1}{n} \left(\prod_{k=1}^{m-1} 1 - \frac{|i-j|}{n+k} \right) S_m^{i-j}.$$

Under summability assumptions, this system is C^* -encoding.

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Under summability assumptions, this system is C^* -encoding.

Remark

A unital CPC^ system is automatically NF. This is not so for C^* -encoding systems.*

C^* -encoding systems

Theorem (C. '23)

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1. *A is nuclear.*
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Moreover, for any nuclear C^ -algebra A and any¹ system*

$(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of cpc approximations of A the induced cpc system $(F_n, \psi_{n+1} \circ \varphi_n)_n$ is C^ -encoding and its limit is coi to A .*

¹after possibly passing to a summable subsystem— same for NF and CPC^*

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Moreover, for any nuclear C^ -algebra A , there exists a system*

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Moreover, for any nuclear C-algebra A, **there exists** a system*

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For a finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$, the following are equivalent

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- 2. The limit is coi to a nuclear C^* -algebra. (CW, OS)*
- 3. The system has a C^* -encoding subsystem.*

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- 3. The system has a C*-encoding subsystem.*

That means C*-encoding is necessary and sufficient to have a limit coi to a (nuclear) C*-algebra, and that the multiplication

$$\rho_k(x) \bullet \rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x) \rho_{n,k}(y)), \quad k \geq 0, x, y \in F_k,$$

is essentially the only possible C*-product on the limit.

Part IV: Nuclear Operator Systems

Non- C^* -encoding systems?

Question

Is there a finite-dimensional cpc system with no C^ -encoding subsystem, i.e., whose limit is not coi to a C^* -algebra?*

C^* -encoding systems (C.)

Definition (C.'23)

A finite-dimensional cpc system $(F_n, \rho_{n+1,n})_n$ is C^* -encoding if for any $k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, there exists an $M > k$ so that for all $m > n, j > M$

$$\|\rho_{m,n}(\rho_{n,k}(x)\rho_{n,k}(y)) - \rho_{m,j}(\rho_{j,k}(x)\rho_{j,k}(y))\| < \varepsilon.$$

A simple example

Example (C.–Galke–van Luijk–Stottmeister)

The finite-dimensional cpc system $(M_n, \rho_{n+1,n})_n$ with

$$\rho_{n+1,n}(y) = y \oplus y_{11}$$

has no C^* -encoding subsystem, and its limit is not coi to a C^* -algebra.

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Note that for $E_{12}, E_{21} \in M_2$ and $n > 2$, $\rho_{n,2}(E_{12}) = E_{12} \in M_n$ and $\rho_{n,2}(E_{21}) = E_{12} \in M_n$.

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$$\rho_{m,n}(\rho_{n,2}(E_{12})\rho_{n,2}(E_{21})) = \rho_{m,n}(E_{11}) = E_{11} \oplus I_{m-n}.$$

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Note that for $E_{12}, E_{21} \in M_2$ and $n > 2$, $\rho_{n,2}(E_{12}) = E_{12} \in M_n$ and $\rho_{n,2}(E_{21}) = E_{12} \in M_n$. So for any $m > n > 2$,

$$\rho_{m,n}(\rho_{n,2}(E_{12})\rho_{n,2}(E_{21})) = \rho_{m,n}(E_{11}) = E_{11} \oplus I_{m-n}.$$

Then for all $m > n > j > 2$

$$\|\rho_{m,n}(\rho_{n,2}(E_{12})\rho_{n,2}(E_{21})) - \rho_{m,j}(\rho_{j,2}(E_{12})\rho_{j,2}(E_{21}))\| = 1.$$

Non- C^* -encoding systems

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Yes. Moreover, since the maps in our example are ucp, the limit is an operator system, which is not coi to a C^* -algebra. Moreover, it is a nuclear operator system.

Nuclear Operator Systems

Theorem (Han–Paulsen, '11)

A separable operator system $\mathcal{S}$ is *nuclear* in the category of operator systems (i.e., *(max,min)-nuclear*) iff there exist ucp maps

$\mathcal{S} \xrightarrow{\psi_n} M_{k_n} \xrightarrow{\varphi_n} \mathcal{S}$ such that $\varphi_n \circ \psi_n \rightarrow \text{id}_{\mathcal{S}}$ pointwise in norm.

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Proposition (C.–Galke–van Luijk–Stottmeister)

Let $\mathcal{S}$ be a separable nuclear operator system and $(\mathcal{S} \xrightarrow{\psi_n} M_{k_n} \xrightarrow{\varphi_n} \mathcal{S})_n$ a system of completely positive approximations. After possibly passing to a summable subsystem, $(M_{k_n}, \psi_{n+1} \circ \varphi_n)_n$ is a cpc system whose limit is coi to $\mathcal{S}$ via the map $a \mapsto [(\psi_n(a))_n]$.

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Hence any separable nuclear operator system is coi to the limit of a finite-dimensional cpc system.

Nuclear Operator Systems not cois to C^* -algebras

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Theorem (C.–Galke–van Luijk–Stottmeister)

Let S be a separable operator system. Then the following are equivalent.

- 1. S is nuclear and completely order isomorphic to a C^* -algebra.*
- 2. S is completely order isomorphic to the limit of a finite-dimensional C^* -encoding system.*

A simple example revisited

Consider the finite-dimensional cpc system $(M_n, \rho_{n+1,n})_n$ with

$$\rho_{n+1,n}(y) = y \oplus y_{11}.$$

Proposition (C.–Galke–van Luijk–Stottmeister + Han–Paulsen)

The limit is coi to the nuclear operator system

$$\mathcal{S} = \overline{\text{span}}\{I, E_{ij} \mid (i,j) \neq (1,1)\} \subset B(\ell^2(\mathbb{N})).$$

With this and the previous theorem, we recover Han and Paulsen's result.

Theorem (Han–Paulsen, '11)

$\mathcal{S}$ is not coi to a C^ -algebra.*

Thank you.

Summability

A system of c.p.c. approximations $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ of a separable C^* -algebra A is **summable** if there exists a decreasing sequence $(\epsilon_n) \in \ell^1(\mathbb{N})_+^1$ so that $\|\varphi_n - \varphi_m \circ \psi_m \circ \varphi_n\| < \epsilon_n$ for all $m > n \geq 0$.

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We will call a Følner sequence $(\mathcal{G}_n)_n$ for a discrete group G **summable** if there exists a decreasing sequence $(\varepsilon_n) \in \ell^1(\mathbb{N})_+^1$ so that for all $m > n \geq 0$

$$\max_{g, h \in \mathcal{G}_n} \left(1 - \frac{|\mathcal{G}_m \cap gh^{-1}\mathcal{G}_m|}{|\mathcal{G}_m|} \right) |\mathcal{G}_n| < \varepsilon_m.$$

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One sub-Følner sequence of $(\{0, \dots, n\})_n$ for $\mathbb{Z}$ making the system of cpc approximations from before summable (for $\varepsilon_n = 2^{n+1}$) is given by $\mathcal{G}_0 = \{0\}$ and $\mathcal{G}_n = \{0, \dots, 2^n |\mathcal{G}_{n-1}|\}$ for $n \geq 1$.

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$$\varphi_n(\psi_n(\lambda_k)) = \varphi_n(S_{|\mathcal{G}_n|}^k) = \frac{|\mathcal{G}_n| - |k|}{|\mathcal{G}_n|} \lambda_k$$

for $n > k \geq 0$ where $S_{|\mathcal{G}_n|} \in M_{|\mathcal{G}_n|}$ is the shift. A few iterations yields

$$\rho_{m,n}(e_{i,j}) = \frac{1}{|\mathcal{G}_n|} \left(\prod_{k=1}^{m-1} \frac{|\mathcal{G}_{n+k}| - |i-j|}{|\mathcal{G}_{n+k}|} \right) S_{|\mathcal{G}_m|}^{i-j}, \quad \text{for } m > n \geq 0, 0 \leq i, j \leq n.$$