C^*-simplicity & Thompson Groups

Based on "Non-inner amenability of the Thompson groups T and V"
Haagerup & Olesen (2016)

Intro

Thompson (1965) defined the groups $F < T < V$ (known now as as the Thompson groups) in a set of handwritten notes. It seems his interest was initially driven by the interest in finding finitely generated groups with unsolvable word problem.

However in 1979, Geoghegan conjectured that F is nonamenable and does not contain a copy of $\mathbb{F}_2$, i.e. that it is a counterexample to the

<u>von Neumann-Day Conjecture</u>

Day noted in 1957 that (using vN 1929), we have the following implications. For discrete gp G. He remarked that the converses were unknown.

G is elementary amenable $\Rightarrow$ G is amenable $\Rightarrow$ G does not contain a copy of $\mathbb{F}_2$

(i.e. G is in the smallest class of gps containing all finite & amenable gps & is closed under subgps, quotients, extensions, & directed unions w.r.t. inclusion)

$\overset{?}{\not\Leftarrow}$ $\overset{?}{\not\Leftarrow}$

Both were later proven false

Olshanskii (1980)

Grigorchuk (1984)

Other examples exist but are rare

Brin-Squire (1985) F does not contain a copy of $\mathbb{F}_2$

Cannon-Floyd-Parry (1996) F is not elementary amenable.

So, the question of the amenability of F is still very interesting and not just enticing!

Remark T does contain a copy of $\mathbb{F}_2$.

The Thompson Groups

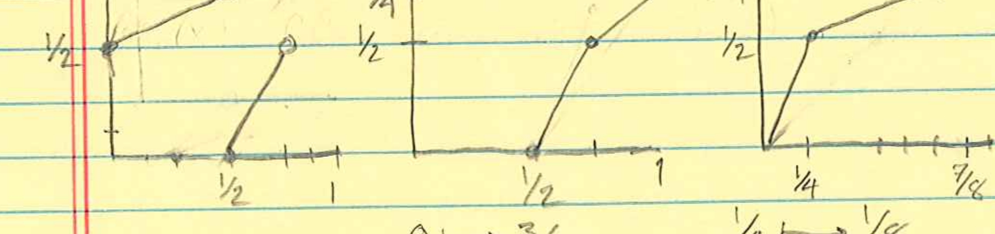
These have numerous realizations, and some facts are more readily proved from different realizations. We will give only one, and so some relevant facts we will just have to quote.
(For more information on these groups, consult Cannon-Floyd-Parry's introductory notes)

Let $\mathcal{X} = \mathbb{Z}[1/2] \cap [0,1)$ denote the standard dyadic rationals on $[0,1)$.

V is the set of bijections on $[0,1)$ that satisfy the following:
1) They are right continuous & piecewise linear
2) They have finitely many points of non-differentiability all of which lie in $\mathcal{X}$.
3) On intervals of differentiability, their derivatives are of the form 2^j, $j \in \mathbb{N}$.

$T = \{ f \in V : f$ has at most one point of discontinuity$\}$
Equivalently, T consists of elements of V which define a homeomorphism of $[0,1)$ when identified with $\mathbb{R}/\mathbb{Z}$ in the standard way.

$F = \{ f \in V : f$ is a homeomorphism of $[0,1)\}$
Equivalently, $F = \{ f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \mid f \in T \text{ \& } f(0) = 0\}$.



$0 \mapsto 3/4$
$1/2 \mapsto 0$

$1/4 \mapsto 1/8$
$7/8 \mapsto 3/4$

$f' > 0$ and f having at most one point of discon[tinuity] forces $f(x_0) = 0$ & $\lim_{x \to x_0} f(x) = 1$ at this one point x_0.

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Remarks

1) These are groups where multiplication is composition.
An inductive argument shows that on an interval of differentiability, any f in V has the form $f(x) = 2^j x + \frac{p}{2^k}$ where $j, k \in \mathbb{N}$ & $p \in \mathbb{Z}$.
So, for any $f \in V$, $f(\mathcal{X}) = \mathcal{X}$ & it follows that V is closed w.r.t. composition & forming inverses.
Taking T to be a set of homeomorphisms on S^1, it follows readily that T & F are also groups.

2) From remark (1), we see that $V \curvearrowright \mathcal{X}$ (and so do T & F) by definition.
This gives us a representation
$$\pi : V \longrightarrow B(\ell^2(\mathcal{X})), \quad \pi(f)\delta_x = \delta_{f(x)}$$

3) $T \curvearrowright \mathcal{X}$ transitively, i.e. for any $x, y \in \mathcal{X}$, $\exists$ $f \in T$ with $f(x) = y$.
This also holds for F when $x, y \neq 0$.
(See given examples

II Main Theorem

(Bleak-Juschenko (2014)
Breuillard-Kalantar-Kennedy-Ozawa (2014)
Haagerup-Olesen (2016)
Le Boudec-Matte Bon (2016))

π_T the induced rep
i.e. $\lambda_T : \mathbb{C}T \to B(\ell^2(T))$ the induced rep.

TFAE

[HO] [HO] [B] [HO] [HO]

1) F is non-amenable
2) $C^*_r(T)$ is simple [BKKO] [LM]
3) $C^*_r(F)$ is simple
4) $\exists\ y \in \mathbb{C}T$ s.t. $\pi|_T(y) = 0$ but $\langle \lambda_T(y) \rangle_{C^*_r(T)} = C^*_r(T)$
5) $\exists\ y \in \mathbb{C}F$ s.t. $\pi|_F(y) = 0$ but $\langle \lambda_F(y) \rangle_{C^*_r(F)} = C^*_r(F)$
6) $C^*_r(F)$ has UTP (i.e. has a unique tracial state)

[?] [BKKO]

We will cover the arguments from [HO]
for $(1) \Leftrightarrow (6)$, $(1) \Rightarrow (5) \Rightarrow (4) \Rightarrow (1)$, & $(2) \Rightarrow (1)$

III Preliminaries

In what follows, G is a discrete group

Weak Containment

Defn Given two unitary representations
$\varphi : G \to \mathcal{U}(\mathcal{H})$ & $\rho : G \to \mathcal{U}(\mathcal{H}')$,
we say φ is weakly contained in ρ, $\varphi \prec \rho$,
if $\exists$ a $*$-hom $\sigma : C^*(\rho(G)) \to C^*(\varphi(G))$ s.t. $\varphi = \sigma \circ \rho$
i.e. $\ker(\rho : C^*(G) \to B(\mathcal{H}')) \subset \ker(\varphi : C^*(G) \to B(\mathcal{H}))$
i.e. $\forall\ y \in \mathbb{C}G$, $\|\varphi(y)\| \le \|\rho(y)\|$

Recall, a discrete group G is amenable
iff the trivial rep $1_G : G \to \mathcal{U}(\ell^2(G))$
is weakly contained in the left-regular rep.

A fact for weak containment

Suppose G acts on a set Y & let $\rho : G \to \mathcal{U}(\ell^2(Y))$
be the induced rep. Then $\rho \prec \lambda$ if all stabilizers
$G_x := \{ g \in G \mid gx = x \}$ are amenable.

Lemma [HO]

F is amenable iff $\overset{\pi|_T}{\overset{\shortparallel}{\pi}} \prec \lambda_T$ iff $\overset{\pi|_F}{\overset{\shortparallel}{\pi}} \prec \lambda_F$.

Proof

Suppose F is amenable.

Since $T \curvearrowright \mathcal{X}$ is transitive, all stabilizers are isomorphic and hence isomorphic to $T_0 = F$. Then, by our fact for weak containment $\pi \prec \lambda_T$.

Since $C^*_r(F) \hookrightarrow C^*_r(T)$ canonically, $\forall\, y \in \mathbb{C}F$ we have $\|\pi(y)\| \leq \|\lambda_T(y)\| = \|\lambda_F(y)\|$

So, $\pi = \pi|_F \prec \lambda_F$.

Now, suppose $\pi \prec \lambda_F$.

Let p denote the projection of $\ell^2(\mathcal{X})$ onto $\mathbb{C}\delta_0$. Since this is a 1-dimensional $\pi(F)$-invariant subspace of $\mathcal{X}$, the map $p\pi p: F \to \mathcal{U}(\ell^2(\mathcal{X}))$ is the trivial rep. of F on $\ell^2(\mathcal{X})$.

Then, for any $y \in \mathbb{C}F$,

$$\|p\pi(y)p\| \leq \|\pi(y)\| \leq \|\lambda_F(y)\|$$

Which means $1_F \prec \lambda_F$ & hence F is amenable

$\square$

IV UTP and Non-amenability of F

(1)⇔(6) Haagerup–Olesen give an argument for F non-amenable ⟺ $C^*_r(F)$ has UTP based on the following theorem of [BKKO] and fact for F.

Thm [BKKO] For a discrete group G, $C^*_r(G)$ has UTP $\Leftrightarrow R(G) = \{0\}$ where R(G) is the max'l amenable normal subgroup of G.

Fact for F (see Cannon–Floyd–Parry) Every normal subgroup of F contains a copy of F

Proof for (1) ⟺ (6)
$R(G) \neq 0 \Rightarrow F \lesssim R(G) \Rightarrow F$ amenable $\Rightarrow R(G) \neq 0$.

The equivalence of (1) & (6) will be useful for the rest of the proof given the following characterization of the UTP due to Haagerup:

Theorem (Haagerup 2015)
For a discrete group G, $C^*_r(G)$ has UTP iff
~~$\forall t \in G\setminus\{e\}$ $0 \in \text{conv}\{\lambda(sts^{-1}) \mid s \in G\}$~~
ie $\forall t \in G\setminus\{e\}$ & $\varepsilon > 0$ $\exists$ $s_1, \ldots, s_n \in G$ s.t.
$$\left\|\frac{1}{n}\sum_{k=1}^{n} \lambda(s_k t s_k^{-1})\right\| < \varepsilon$$

Compare to Power's Test for C^*-simplicity:
(Kennedy 2015, Haagerup 2015)
$C^*_r(G)$ is simple iff for any $t_1, \ldots, t_m \in G\setminus\{e\}$ & $\varepsilon > 0$
$\exists$ $s_1, s_2, \ldots, s_n \in G$ s.t. $\left\|\frac{1}{n}\sum_{k=1}^{n} \lambda(s_k t_j s_k^{-1})\right\| < \varepsilon$ for $j = 1, \ldots, m$
(ie for any $a \in C^*_r(G)$ & $\varepsilon > 0$ $\exists$ $s_1, \ldots, s_n \in G$ s.t.
$$\left\|\frac{1}{n}\sum_{k=1}^{n} \lambda_{s_k} a \lambda_{s_k}^{-1} - \tau_\lambda(a) 1\right\| < \varepsilon$$)

VI Main Proof

$(1) \Rightarrow (5)$

Assume F is nonamenable; we want $y \in \mathbb{C}F$ s.t. $\pi(y) = 0$ & $\langle \lambda_F(y) \rangle_{C^*(F)} = C^*_r(F)$

Let $F_1 = \{ f \in F \mid f|_{[0,1/2)} = e \}$ & $F_2 = \{ f \in F \mid f|_{[1/2,1)} = e \}$

Choose any $g \in F_1 \setminus \{e\}$ and $h \in F_2 \setminus \{e\}$

Then $gh(x) = hg(x) = \begin{cases} h(x), & x \in [0, 1/2) \\ g(x), & x \in [1/2, 1) \end{cases}$

In particular, for any $x \in \mathcal{X}$,

$\{g(x), h(x)\} = \{gh(x), x\}$

Let $y = 1 + gh - g - h \in \mathbb{C}F$.

Then for any $x \in \mathcal{X}$, $\pi(1+gh)\delta_x = \pi(g+h)\delta_x$, and so $\pi(y) = 0$.

It remains to show $C^*_r(F) = \langle \lambda(y) \rangle$, ie. $1 \in \langle \lambda(y) \rangle$.

Since $\overline{\mathrm{conv}}\{\lambda(s)(1 + \lambda(gh) - \lambda(g) - \lambda(h))\lambda(s^{-1}) \mid s \in G\}$
$= 1 + \overline{\mathrm{conv}}\{\lambda(sghs^{-1}) + \lambda(sgs^{-1}) - \lambda(shs^{-1}) \mid s \in G\}$
$=: 1 + K \subseteq \langle \lambda(y) \rangle$

It suffices to show $0 \in K$.

Since F is assumed non-amenable and $F \cong F_1 \cong F_2$, it follows from $(1) \Leftrightarrow (6)$ that F_1 & F_2 have the UTP. By Haagerup's Theorem, this means for any $\varepsilon > 0$,

$\exists\, s_1, \dots, s_n \in F_1$ & $t_1, \dots, t_m \in F_2$ s.t.

Using $C^*_r(F_1) \subseteq C^*_r(F)$ canonically

$\| \underbrace{\tfrac{1}{n} \textstyle\sum_{i=1}^n \lambda(s_i g s_i^{-1})}_{=:a} \| < \varepsilon$ & $\| \underbrace{\tfrac{1}{m} \textstyle\sum_{j=1}^m \lambda(t_j h t_j^{-1})}_{=:b} \| < \varepsilon$

Now, since $[F_1, F_2] = \{0\}$, we compute

$\frac{1}{nm} \sum_{i=1}^n \sum_{j=1}^m \lambda(s_i t_j)(\lambda(gh) - \lambda(g) - \lambda(h))\lambda(s_i t_j)^{-1}$
$= \frac{1}{n}\sum_{i=1}^n \frac{1}{m}\sum_{j=1}^m \lambda(s_i t_j g h t_j^{-1} s_i^{-1})$
$- \frac{1}{n}\sum_{i=1}^n \frac{1}{m}\sum_{j=1}^m \lambda(s_i t_j g t_j^{-1} s_i^{-1}) - \frac{1}{n}\sum_{i=1}^n \frac{1}{m}\sum_{j=1}^m \lambda(s_i t_j h t_j^{-1} s_i^{-1})$
$= \frac{1}{n}\sum_{i=1}^n \frac{1}{m}\sum_{j=1}^m \lambda(s_i g s_i^{-1})\lambda(t_j h t_j^{-1})$
$- \frac{1}{n}\sum_{i=1}^n \frac{1}{m}\sum_{j=1}^m \lambda(s_i g s_i^{-1}) - \frac{1}{n}\sum_{i=1}^n \frac{1}{m}\sum_{j=1}^m \lambda(t_j h t_j^{-1})$
$= ab - a - b$

So, $ab - a - b \in K$ and $\|ab - a - b\| < \varepsilon^2 + 2\varepsilon$

Since ε was arbitrary, we conclude that $0 \in K$.

$\boxed{(5) \Rightarrow (4)}$

$\$\ \exists\, y \in \mathbb{C}F$ s.t. $\pi(y) = 0$ but $\langle \lambda_F(y) \rangle_{C^*_r(F)} = C^*_r(F)$.
We claim $\langle \lambda_T(y) \rangle_{C^*_r(T)} = C^*_r(T)$.
Since $C^*_r(F) \subset C^*_r(T)$ is a unital embedding, we have $1_T \in \langle \lambda_F(y) \rangle_{C^*_r(F)} \subset \langle \lambda_T(y) \rangle_{C^*_r(T)}$

$\boxed{(4) \Rightarrow (1)}$

i.e. $\exists\, y \in \mathbb{C}T$ s.t. $\pi(y) = 0$ & $\langle \lambda(y) \rangle_{C^*_r(T)} = C^*_r(T)$
$\Longrightarrow$ F non-amenable.

Assume F is amenable.
By our weak containment lemma, this means $\pi \prec \lambda_T$, which means $\exists$ a nonzero $*$-hom $\sigma : C^*_r(T) \to C^*(\pi(T))$ s.t. $\pi = \sigma \circ \lambda$.
$\$\ y \in \mathbb{C}T \setminus \{0\}$ with $\pi(y) = 0$. Then $\lambda(y) \neq 0$, and hence $\lambda(y) \in \ker(\sigma)$, which means $\ker(\sigma) \neq \{0\}$
Thus, $\ker(\sigma)$ is a nontrivial proper ideal, and hence so is $\langle \lambda(y) \rangle_{C^*_r(T)} \subseteq \ker(\sigma)$.

$\square$

We are actually quite close to showing
T C^*-simple $\Rightarrow$ F non-amenable.

Proof

Assuming F is amenable, from the $(4) \Rightarrow (1)$ proof above, we see that we need only exhibit a $y \in \mathbb{C}T \setminus \{0\}$ such that $\pi(y) = 0$ to arrive at a proper ideal of $C^*_r(T)$.
But we showed in $(1) \Rightarrow (5)$ how to construct many such elements.

$\square$

Extra Remarks

1) [LM] Prove (1) $\Rightarrow$ (3) by characterizing the uniformly recurrent subgroups of F.
They show these are exactly the normal sbgps
Recall from Sabrina's talk that
(Kennedy) a discrete gp G is C^*-simple iff it has no nontrivial amenable URS
Recall our Fact for F that every nontrivial normal sbgp of F contains a copy of F.
So, F not C^*-simple $\Rightarrow$ F amenable

2) [BKKO] Proof (3) $\Rightarrow$ (2) utilizes their proposition that for a discrete group G with G-boundary X, if $\exists\, x \in G$ w/ G_x C^*-simple, then G is C^*-simple (Recall $T_0 = F$)

3) [LM] Prove V is C^*-simple

Recommended Reading

[BJ] Bleak, Juschenko (2014) ([1])
[BKKO] Breuillard, Kalantar, Kennedy, Ozawa (2017) ([2])
[CFP] Cannon, Floyd, Parry, "Introductory notes on Richard Thompson's Groups." (1996)
[H] Haagerup (2016) ([5])
[HO] Haagerup, Olesen (2017) ([6])
[LM] Le Boudec, Matte Bon (2016) ([14])