Generalized inductive limits of C^*-algebras

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Inductive limits of C^*-algebras

An inductive system of C^*-algebras is a sequence $(A_n)_n$ of C^*-algebras together with coherent connecting *-homomorphisms $\rho_{m,n} : A_n \to A_m$

$$A_0 \xrightarrow{\rho_{1,0}} A_1 \xrightarrow{\rho_{2,1}} A_2 \longrightarrow \dots \qquad (\rho_{2,0} : A_0 \to A_2)$$

For each $k \geq 0$ and $a \in A_k$, we get a norm-bounded sequence

$$(\rho_{n,k}(a))_n \in \prod_n A_n.$$

The quotient map $\prod_n A_n \to {}^{\prod A_n}/_{\bigoplus A_n}$ induces *-homomorphisms $\rho_k : A_k \to {}^{\prod A_n}/_{\bigoplus A_n}$ for each $k \geq 0$ by

$$\rho_k(a) = \big[(\rho_{n,k}(a))_n\big], \ \forall\ a \in A_k.$$

The inductive limit of the system $(A_n, \rho_{m,n})$ is the C^*-algebra

$$\varinjlim(A_n, \rho_{m,n}) := \overline{\bigcup_k \rho_k(A_k)} \subset {}^{\prod A_n}/_{\bigoplus A_n}.$$

Inductive limits of C^*-algebras

This inductive limit construction has provided many interesting examples of C^*-algebras, in particular, the AF algebras.

Definition

A C^*-algebra is Approximately Finite Dimensional (AF) if it is *-isomorphic to the inductive limit of finite-dimensional C^*-algebras.

Example

The compact operators

$$\mathbb{C} \hookrightarrow \mathrm{M}_2 \xhookrightarrow{a \mapsto a \oplus 0} \mathrm{M}_3 \hookrightarrow \mathrm{M}_4 \hookrightarrow \dots \longrightarrow \mathcal{K}(\ell^2)$$

The CAR algebra:

$$\mathbb{C} \hookrightarrow \mathrm{M}_2 \xhookrightarrow{a \mapsto a \oplus a} \mathrm{M}_4 \hookrightarrow \mathrm{M}_8 \hookrightarrow \dots \longrightarrow \mathrm{M}_{2^\infty}$$

Nuclear C^*-algebras

Theorem/Definition (Choi–Effros '78; Kirchberg '77)

A separable C^*-algebra A is nuclear iff there exists a sequence of finite-dimensional C^*-algebras $(F_n)_{n\in\mathbb{N}}$ and completely positive contractive (cpc) maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ such that for all $a \in A$

$$\|\varphi_n\big(\psi_n(a)\big) - a\| \to 0.$$

We often think of a sequence of approximately commuting diagrams.

$$\begin{array}{ccccccccccc}
A & & \xrightarrow{\quad\mathrm{id}\quad} & & A & & \xrightarrow{\quad\mathrm{id}\quad} & & A & \xrightarrow{\quad\mathrm{id}\quad} & \dots \\
& \searrow^{\psi_0} & & \nearrow_{\varphi_0} & & \searrow^{\psi_1} & & \nearrow_{\varphi_1} & & \searrow^{\psi_2} \quad \nearrow_{\varphi_2} & \\
& & F_0 & & & & F_1 & & & F_2 & \dots
\end{array}$$

We call $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ a system of cpc approximations of A.

Example

Any AF C^*-algebra is nuclear. But most nuclear C^*-algebras are not AF.

More?

To build more nuclear C^*-algebras as inductive limits, one route is to make the building blocks more sophistocated:

Definition

A C^*-algebra is Approximately (Sub)Homogeneous (A(S)H) if it is *-isomorphic to the inductive limit of (sub)homogeneous C^*-algebras.

Another route is to make the maps $\rho_{m,n} : A_n \to A_m$ less rigid.

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Another route is to make the maps $\rho_{m,n} : A_n \to A_m$ less rigid.

(Sub)homogeneous means $\cong \bigoplus_{k=1}^{m} M_{n_k}(C_0(X_k))$.

Generalized inductive limits

Definition (Blackadar-Kirchberg)

A generalized inductive system $(A_n, \rho_{m,n})$ consists of a sequence $(A_n)_n$ of C^*-algebras and coherent maps $\rho_{m,n} : A_n \to A_m$ that are

1. Pointwise bounded: $\sup_m \|\rho_{m,n}(x)\| < \infty, \ \forall \ n \geq 0, x \in A_n$,
2. Asymptotically *-linear, and
3. Asymptotically multiplicative:
   For any $\varepsilon > 0$, $k \geq 0$, $x, y \in A_k$, there exists $M > k$ such that for all $m > n > M$

$$\|\rho_{m,n}(\rho_{n,k}(x)\rho_{n,k}(y)) - \rho_{m,k}(x)\rho_{m,k}(y)\| < \varepsilon.$$

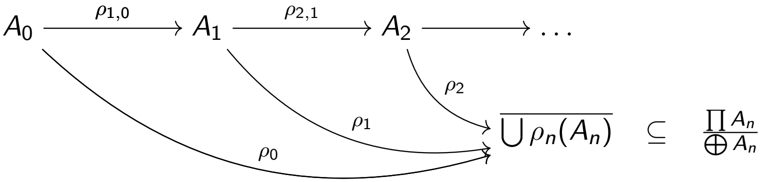
Think of this as saying that the maps $\rho_{m,n}$ become more multiplicative on $\rho_{n,k}(x)$ and $\rho_{n,k}(y)$.

$$\|\rho_{m,n}(\rho_{n,k}(x)\rho_{n,k}(y)) - \rho_{m,n}(\rho_{n,k}(x))\rho_{m,n}(\rho_{n,k}(y))\| < \varepsilon.$$

Generalized inductive limits

We form the limit in the same way:

Just as before we have induced maps $\rho_k : A_k \to \prod A_n / \bigoplus A_n$, now bounded and *-linear, given by $\rho_k(x) = [(\rho_{m,k}(x))]$.



The limit, $\varinjlim(A_n, \rho_{m,n}) := \overline{\bigcup \rho_n(A_n)}$, is a closed self adjoint subspace, and is moreover closed w.r.t. multiplication: indeed for $n \geq 0, x, y \in A_k$,

$$\rho_k(x)\rho_k(y) = \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y))$$

And hence the limit is a C^*-algebra.

MF Algebras

Theorem (Blackadar-Kirchberg)

The following are equivalent for a separable C^*-algebra A:

1. *A can be written as the limit of a generalized inductive system of finite-dimensional C^*-algebras.*
2. *There exists a $*$-isomorphic embedding $A \hookrightarrow \prod M_{k_n} / \bigoplus M_{k_n}$ for some sequence $(k_n)_n$.*
3. *A admits norm microstates.*

Definition

A separable C^*-algebra is called MF if it satisfies any of the above.

Example

1. Any separable quasidiagonal (QD) C^*-algebra is MF.
2. $\mathrm{C}^*_r(\Gamma)$ where $\Gamma = \mathbb{F}_2$ (Haagerup–Thorbjørnsen), $\Gamma = G_1 * G_2$ with G_i MF (Hayes), $\Gamma = G_1 *_H G_2$ with G_i amenable (Schafhauser), $\Gamma = G \rtimes \mathbb{F}$ with G amenable (Rainone–Schafhauser), $\Gamma =$ limit group (Louder–Magee), $\Gamma =$ right-angle Artin group (Magee–Thomas).

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$A = \mathrm{C}^*(x_1, \dots, x_s)$ $(x_i = x_i^*)$ admits norm microstates if for every finite set $\mathcal{F}$ of noncommutative polynomials in s variables and $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and self-adjoint matrices $X_1, \dots, X_s \in \mathrm{M}_n$ such that

$$\max_{P \in \mathcal{F}} \left| \|P(x_1, \dots, x_s)\| - \|P(X_1, \dots, X_s)\| \right| < \varepsilon.$$

More nuclear C^*-algebras

Theorem (Blackadar-Kirchberg)

The following are equivalent for a separable C^*-algebra A:

1. *A can be written as the limit of a generalized inductive system of finite-dimensional C^*-algebras with cpc connecting maps.*
2. *A is nuclear and MF.*
3. *A is nuclear and QD.*
4. *A admits a system of completely positive approximations $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ with ψ_n approximately multiplicative.*

Definition

A separable C^*-algebra is called NF if it satisfies any of the above.

Example

$\mathrm{C}^*_r(\Gamma)$ where Γ is any discrete amenable group (Tikuisis–White–Winter).
Any ASH C^*-algebra (BK)

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Definition

$$\|\psi_n(ab) - \psi_n(a)\psi_n(b)\| \to 0, \ \forall\, a, b \in A$$

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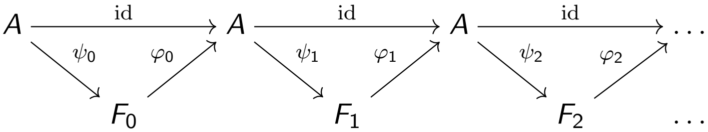
From CPAP to an NF system

Recall that a system of cpc approximations $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ of a nuclear C^*-algebra A yields a sequence of approximately commuting diagrams:



Proposition (BK, C.)

Taking $\rho_{m,n} = \psi_m \circ \varphi_{m-1} \circ \ldots \circ \varphi_n$ *gives*[†] *an NF system iff the* ψ_n *are approximately multiplicative.*

And the limit is isomorphic to A*.*

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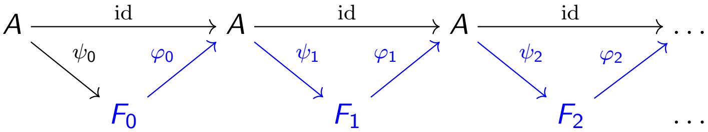

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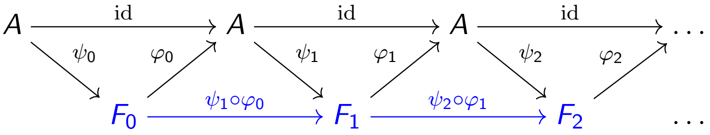

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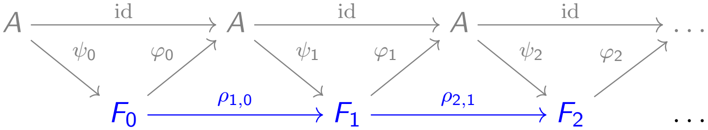

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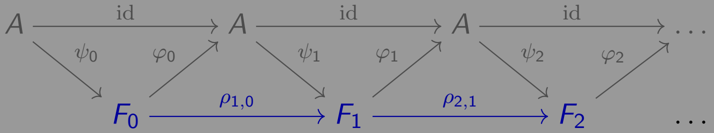

Proposition (BK, C.)

Taking $\rho_{m,n} = \psi_m \circ \varphi_{m-1} \circ \ldots \circ \varphi_n$ gives† an NF system iff the ψ_n are approximately multiplicative.

And the limit is isomorphic to A.

† First you have to pass to a "summable" subsystem of approximations (which one can always do) so that $\lim_{m \gg n} \|\rho_{m,n} - \psi_m \circ \varphi_n\| = 0$.

From CPAP to an NF system

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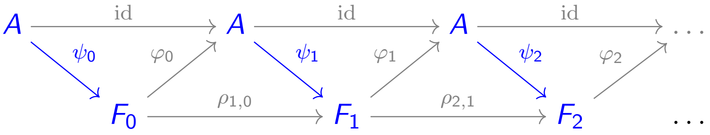

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Taking $\rho_{m,n} = \psi_m \circ \varphi_{m-1} \circ \ldots \circ \varphi_n$ gives rise to† *an NF system iff the ψ_n are approximately multiplicative.*

And the limit is isomorphic to A via $a \mapsto [(\psi_n(a))_n]$.

More nuclear C^*-algebras

Theorem (Blackadar-Kirchberg)

The following are equivalent for a separable C^*-algebra A:

1. *A can be written as the limit of a generalized inductive system (called an NF system, of finite-dimensional C^*-algebras with $\rho_{m,n}$ cpc.*
2. *A is nuclear and MF.*
3. *A is nuclear and QD.*
4. *A admits a system of completely positive approximations $(A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A)_n$ with ψ_n approximately multiplicative.*

Definition

A separable C^*-algebra is called NF if it satisfies any of the above.

Example

$\mathrm{C}^*_r(\Gamma)$ where Γ is any discrete amenable group (Tikuisis–White–Winter).
Any ASH C^*-algebra (BK)

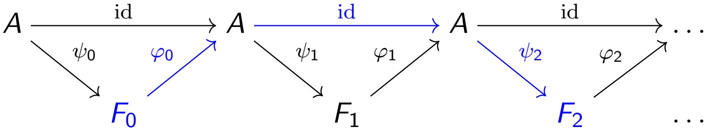
Soft Inductive Systems

Definition (van Luijk–Stottmeister–Werner, 23)

A soft inductive sequence of C^*-algebras consists of a sequence of C^*-algebras $(A_n)_n$ with *asymptotically coherent* ucp $\rho_{m,n} : A_n \to A_m$.

Example

Given a system $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ of cpc approximations of a nuclear C^*-algebra A, $(A_n, \psi_m \circ \varphi_n)$ forms a soft inductive system.



The sequences are easier to build, but the limits of finite-dimensional systems are still NF.

QDQ

Definition

A C^*-algebra is (stably) finite if it (and any matrix amplification) contains no proper isometry, i.e., $x^*x = 1 \neq xx^*$.

Exercise

Any QD C^*-algebra (in fact any MF algebra) is stably finite.

Remark

That means no infinite (= not finite) C^*-algebra (like $\mathcal{O}_n, \mathcal{T}$, many graph C^*-algebras, etc.) can be NF or even MF.

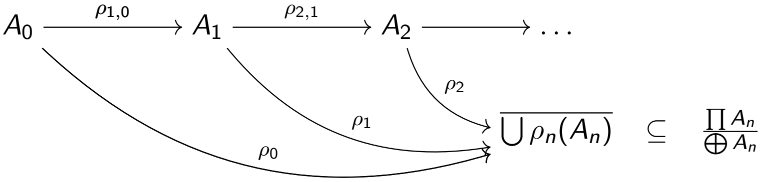
Question (QDQ)

Are all stably finite nuclear C^*-algebras NF?

Inductive systems beyond stably finite

To go beyond stable finiteness, we need to drop asymptotic multiplicativity.

Consider a sequence $(A_n)_n$ of C^*-algebras with coherent cpc connecting maps $\rho_{m,n} : A_n \to A_m$. We can still form the limit as before:



But now the limit, $\varinjlim(A_n, \rho_{m,n}) := \overline{\bigcup \rho_n(A_n)}$ is only an operator system.

Question

When is the limit a C^*-algebra?

Define "is"

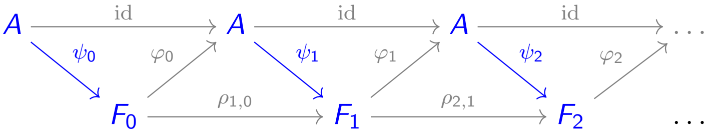
We say an operator system $\mathcal{S}$ is a C^*-algebra if it is completely order isomorphic to a C^*-algebra, i.e., there exists a C^*-algebra A and a completely isometric cp map $\varphi : A \to \mathcal{S}$ with cp inverse.

Remark

1. *We can define a product on $\mathcal{S}$ by $\varphi(x) \bullet \varphi(y) := \varphi(xy)$ with respect to which $\mathcal{S}$ becomes a C^*-algebra.*
2. *A complete order isomorphic between two C^*-algebras is automatically a *-isomorphism.*

Example from CPAP

Recall that a system of cpc approximations $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ of a nuclear C^*-algebra A yields a sequence of approximately commuting diagrams:



The map $\Psi : A \to \prod A_n / \bigoplus A_n$ given by

$$a \mapsto [(\psi_n(a))_n]$$

is a complete order isomorphism onto its image.

A non-example

Given a sequence of C^*-algebras $(A_n)_n$ with cpc connecting maps $\rho_{m,n} : A_n \to A_m$, the limit may not be a C^*-algebra:

Example (Han–Paulsen, C.–Galke–van Luijk–Stottmeister)

The coherent system $(\mathrm{M}_n, \rho_{m,n})$ with $\rho_{n+1,n}(y) = y \oplus y_{11}$ converges to the operator system

$$\mathcal{S} = \overline{\mathsf{span}}\{I, E_{i,j} \mid (i,j) \neq (1,1)\} \subset B(\ell^2(\mathbb{N})),$$

which is not completely order isomorphic to a C^*-algebra.

Beyond Asymptotic Multiplicativity

Let $(A_n)_n$ be a sequence of C^*-algebras with coherent cpc maps $\rho_{m,n} : A_n \to A_m$.

Definition (C.)

The system is C^*-encoding if for any $k \geq 0$, $x, y \in A_k$,

$$\lim_{m \gg n,j} \|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,j}\big(\rho_{j,k}(x)\rho_{j,k}(y)\big)\| = 0.$$

This also guarantees that the $\lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y))$ exists.

This limit still gives a product on the limit $\overline{\bigcup \rho_n(A_n)}$:

$$\rho_k(x) \bullet \rho_k(y) := \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)).$$

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This also guarantees that the $\lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y))$ exists.

This limit still gives a product on the limit $\overline{\bigcup \rho_n(A_n)}$:

Though it may no longer equal $\rho_k(x)\rho_k(y)$.

$$\rho_k(x) \bullet \rho_k(y) := \lim_n \rho_n(\rho_{n,k}(x)\rho_{n,k}(y)).$$

And this product still makes $\overline{\bigcup \rho_n(A_n)}$ a C^*-algebra,

which is completely order isomorphic to $\overline{\bigcup \rho_n(A_n)}$.

Beyond Asymptotic Multiplicativity

Let $(A_n)_n$ be a sequence of C^*-algebras with coherent cpc maps $\rho_{m,n} : A_n \to A_m$.

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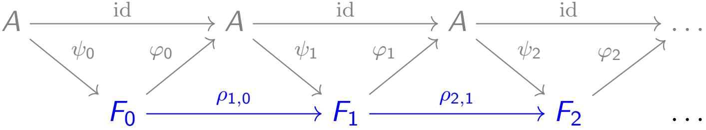
It just might not be the same product as $\prod A_n / \bigoplus A_n$.

which is completely order isomorphic to $\overline{\bigcup \rho_n(A_n)}$.

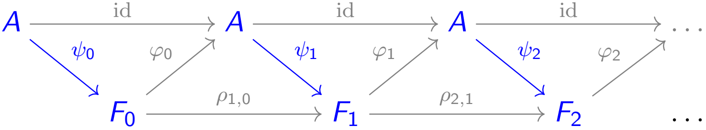
From CPAP to a C^*-encoding system

Theorem (C.)

Any[†] *system of cpc approximations* $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ *of a separable nuclear* C^**-algebra* A *gives rise to a* C^**-encoding system* $(F_n, \psi_m \circ \varphi_{m-1} \circ \ldots \varphi_n)$.



And the limit is completely order isomorphic to A *via* $a \mapsto [(\psi_n(a))_n]$.



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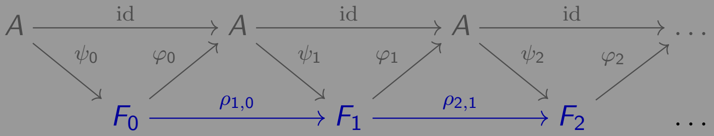

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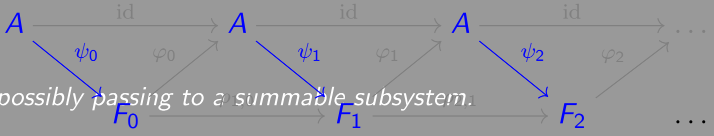

† *After possibly passing to a summable subsystem.*

Example $\mathrm{C}_r^*(\mathbb{Z})$

Given a Følner sequence $(\mathcal{F}_n)_n$ for $\mathbb{Z}$, we have have a system of cpc approximations $\mathrm{C}_r^*(\mathbb{Z}) \xrightarrow{\psi_n} M_{|\mathcal{F}_n|}(\mathbb{C}) \xrightarrow{\varphi_n} \mathrm{C}_r^*(\mathbb{Z})$ with

$$\psi_n\left(\sum_{k\in\mathbb{Z}} a_k\lambda_k\right) = \psi_n\left(\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_0 & a_{-1} & a_{-2} & \ddots \\ \ddots & a_1 & a_0 & a_{-1} & \ddots \\ \ddots & a_2 & a_1 & a_0 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}\right) = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{-(n-1)} \\ a_1 & a_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & \cdots & \cdots & a_0 \end{bmatrix}.$$

and $\varphi_n(e_{i,j}) = \frac{1}{|\mathcal{F}_n|}\lambda_{i-j}$.

When $(\mathcal{F}_n)_n$ is "summable" we get a C^*-encoding system by composing:

$$\rho_{m,n}(e_{i,j}) = \frac{1}{|\mathcal{F}_n|}\prod_{k=1}^{m-1}\frac{|\mathcal{F}_{n+k}| - |i-j|}{|\mathcal{F}_{n+k}|} S_{|\mathcal{F}_n|}^{i-j}.$$

Nuclearity and C^*-encoding systems

So any separable nuclear C^*-algebra is the limit of a finite-dimensional C^*-encoding system. And the converse holds too:

Theorem (Ozawa–Sato, C.–Winter, C.)

Let $(F_n, \rho_{m,n})$ be a finite-dimensional C^*-encoding system. Then the limit is completely order isomorphic to a nuclear C^*-algebra.

Corollary (C.)

The following are equivalent for a separable C^*-algebra A:

1. *A is nuclear.*
2. *A is completely order isomorphic to the limit of a C^*-encoding system.*

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The following are equivalent for a separable C^*-algebra A:

That includes the infinite ones.

1. *A is nuclear.*
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Back to the question

Question

Given a sequence $(A_n)_n$ of C^*-algebras and cpc connecting maps $\rho_{m,n} : A_n \to A_m$, when is the limit completely order isomorphic to a C^*-algebra?

Necessity and Sufficiency

Theorem (C.)

Let $(F_n)_n$ be a sequence of finite-dimensional C^*-algebras with coherent cpc connecting maps $\rho_{m,n} : F_n \to F_m$. TFAE:

1. *The system has a C^*-encoding subsystem.*
2. *The limit is completely order isomorphic to a C^*-algebra.*
3. *The limit is completely order isomorphic to a nuclear C^*-algebra.*

Remark

In spirit, (the proof of) this and the previous theorem establish a 1-1 correspondence between systems of cpc approximations and finite-dimensional C^*-encoding systems.

Nuclear Operator Systems

Definition/Theorem (Han–Paulsen)

A separable operator system $\mathcal{S}$ is nuclear iff there exists a sequence $(k_n)_n$ and cpc maps $\mathcal{S} \xrightarrow{\psi_n} M_{k_n} \xrightarrow{\varphi_n} \mathcal{S}$ such that $\varphi_n \circ \psi_n \to \mathrm{id}_{\mathcal{S}}$ pointwise in order norm.

Theorem (Han–Paulsen)

The operator system

$$\mathcal{S}_0 = \overline{span}\{I, E_{i,j} \mid (i,j) \neq (1,1)\} \subset B(\ell^2(\mathbb{N}))$$

is nulear and not completely order isomorphic to a C^**-algebra.*

Nuclear Operator Systems that are C^*-algebras

Theorem (C.–Galke–van Luijk–Stottmeister)

Let $\mathcal{S}$ be a separable operator system. Then the following are equivalent.

1. *$\mathcal{S}$ is nuclear and completely order isomorphic to a C^*-algebra.*
2. *$\mathcal{S}$ is completely order isomorphic to the limit of a finite-dimensional C^*-encoding system.*

Non-example

Example (C.–Galke– van Luijk–Stottmeister, Han–Paulsen)

The system $(\mathrm{M}_n, \rho_{m,n})_n$ with

$$\rho_{n+1,n}(y) = y \oplus y_{11}$$

is not C^*-encoding. Moreover, it has no C^*-encoding subsystem, and hence its limit,

$$\mathcal{S}_0 = \overline{\mathrm{span}}\{I, E_{i,j} \mid (i,j) \neq (1,1)\} \subset B(\ell^2(\mathbb{N})),$$

is not not completely order isomorphic to a C^*-algebra.

Epilogue: CPC^*-systems

Definition (C.–Winter)

Let $(A_n)_n$ be a sequence of unital C^*-algebras with cpc maps $\rho_{m,n} : A_n \to A_m$. We say $(A_n, \rho_{m,n})$ is a CPC^*-system if the maps are coherent and asymptotically order zero.

Theorem (C.–Winter)

The following are equivalent for a separable C^*-algebra A:

1. *A is nuclear.*
2. *A is completely order isomorphic to the limit of a CPC^*-system.*

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Theorem (C.–Winter)

The following are equivalent for a separable C^*-algebra A:

1. *A is nuclear.*
2. *A is completely order isomorphic to the limit of a CPC^*-system.*

[Winter–Zacharias] A cp map $\psi : A \to B$ from a unital C^*-algebra is order zero iff $\psi(a)\psi(b) = \psi(1)\psi(ab)$ for all $a, b \in A$.

Thank you!