A von Neumann-type inequality with universal C^*-algebras

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von Neumann's Inequality

Theorem (von Neumann, 1951)

Let T be an operator on a Hilbert space with $\|T\| \leq 1$. Then for any polynomial $p \in \mathbb{C}[z]$,

$$\|p(T)\| \leq \sup_{z \in \overline{\mathbb{D}}} |p(z)|,$$

where $\mathbb{D}$ denotes the unit disk of $\mathbb{C}$.

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Theorem (Andô, 1963)

Let T_1 and T_2 be commuting contractions on a Hilbert space, and let $p \in \mathbb{C}[z_1, z_2]$ be a polynomial in two variables. Then

$$\|p(T_1, T_2)\| \leq \sup_{z_1, z_2 \in \overline{\mathbb{D}}} |p(z_1, z_2)|.$$

A $*$-polynomial analogue?

Can we have a similar inequality if p were replaced with a non-commutative $*$-polynomial? For example,

$$q(x) = xx^* - x^*x.$$

A ∗-polynomial analogue?

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To help us see what would be an appropriate analogue, let's first rephrase the original inequality in terms of a universal contraction operator.

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It has the defining property that, given any contraction operator $y \in B(\mathcal{H})$, the assignment $x \mapsto y$ induces a surjective unital $*$-homomorphism

$$C_u^*\langle x : \|x\| \leq 1\rangle \to C^*(y, 1_{\mathcal{H}}).$$

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We call the generator, as well as any image under any faithful nondegenerate representation, a universal contraction.

von Neumann's Inequality, once more

Given any contractive Hilbert space operator T, any polynomial $p \in \mathbb{C}[z]$, and a universal contraction x, we have

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So, von Neumann's inequality can be restated as follows.

Theorem (von Neumann's Inequality)

Let x be a universal contraction operator on some Hilbert space, and let $p \in \mathbb{C}[z]$. Then,

$$\|p(x)\| = \sup_{z \in \overline{\mathbb{D}}} |p(z)|.$$

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So, von Neumann's inequality can be restated as follows.

Theorem (von Neumann's Inequality)

Let x denote the generator of $C^*_u\langle x : \|x\| \leq 1\rangle$, and let $p \in \mathbb{C}[z]$. Then,

$$\|p(x)\| = \sup\{\|p(\pi(x))\| : \pi : C^*_u\langle x : \|x\| \leq 1\rangle \to \mathbb{C}\}.$$

von Neumann's inequality for $*$-polynomials

Actually, given any contractive Hilbert space operator T, any noncommutative $*$-polynomial q, and a universal contraction x, we have

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But the previous theorem fails for noncommutative $*$-polynomials, such as $q(x) = xx^* - x^*x$.

In fact, this theorem will fail for noncommutative $*$-polynomials if we restrict to representations of any bounded finite dimension. This follows from the fact that $C^*_u\langle x : \|x\| \leq 1\rangle$ is not subhomogeneous.

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So, for any noncommutative ∗-polynomial q,

$$\|q(x)\| = \sup_{\pi\in\mathcal{F}} \|\pi(q(x))\| = \sup_{\pi\in\mathcal{F}} \|q(\pi(x))\|.$$

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Proposition (von Neumann's Inequality for $*$-polynomials)

Let x denote the generator of $C_u^*\langle x : \|x\| \leq 1\rangle$, and let q be any noncommutative $*$-polynomial. Then,

$$\|q(x)\| = \sup_n \{\|q(a)\| : a \in M_n(\mathbb{C}), \|a\| \leq 1\}.$$

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Theorem (C., 2017)

Let x denote the generator of $C_u^*\langle x : \|x\| \leq 1\rangle$, and let q be any noncommutative $*$-polynomial with longest monomial of length ℓ. Then,

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Finite-Dimensional Nilpotents

To prove that it suffices to consider finite-dimensional nilpotents, we demonstrate a separating family of finite-dimensional representations of $C_u^*\langle x : \|x\| \leq 1\rangle$, each mapping the generator to a nilpotent operator.

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First, let's build the family of representations.

Universal Nilpotent Contraction C^*-algebras

We denote the universal unital C^*-algebra generated by a contraction satisfying the algebraic relation $x^n = 0$ by

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Theorem (Shulman, 2008)

For each $n \in \mathbb{N}$, $C_u^*\langle x_n : \|x_n\| \leq 1, x_n^n = 0\rangle$ is RFD.

A family of finite-dimensional nilpotent representations

For each $n \in \mathbb{N}$, let $\{\rho_{n,k}\}_{k \in \mathbb{N}}$ be a separating family of finite-dimensional representations for $C^*_u\langle x_n : \|x_n\| \leq 1, x_n^n = 0\rangle$, and let

$$\phi_n : C^*_u\langle x : \|x\| \leq 1\rangle \to C^*_u\langle x_n : \|x_n\| \leq 1,\ x_n^n = 0\rangle$$

be the $*$-homomorphisms induced by mapping $x \mapsto x_n$.

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be the ∗-homomorphisms induced by mapping $x \mapsto x_n$.

$$C_u^*\langle x : \|x\| \leq 1\rangle \xrightarrow{\phi_n} C_u^*\langle x_n : \|x_n\| \leq 1,\ x_n^n = 0\rangle \xrightarrow{\rho_{n,k}} C^*(\rho_{n,k}(x_n), 1_{\mathcal{H}_{\rho_{n,k}}})$$

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$$C^*_u\langle x : \|x\| \leq 1\rangle \xrightarrow{\phi_n} C^*_u\langle x_n : \|x_n\| \leq 1,\ x_n^n = 0\rangle \xrightarrow{\rho_{n,k}} C^*(\rho_{n,k}(x_n), 1_{\mathcal{H}_{\rho_{n,k}}})$$

Then $\rho_{n,k} \circ \phi_n(x) = \rho_{n,k}(x_n)$ is finite-dimensional and nilpotent for all $n, k \in \mathbb{N}$.

A separating family

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Any faithful, unital representation $(\pi, \mathcal{H})$ *of* $C_u^*\langle x : \|x\| \leq 1\rangle$ *on a separable Hilbert space asymptotically factorizes through* $\{\phi_n\}$.

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Theorem (C.,2017)

Any faithful, unital representation $(\pi, \mathcal{H})$ of $C^*_u\langle x : \|x\| \leq 1\rangle$ on a separable Hilbert space asymptotically factorizes through $\{\phi_n\}$.

In other words, there exist $*$-homomorphisms

$$\psi_n : C^*_u\langle x_n : \|x_n\| \leq 1,\ x_n^n = 0\rangle \to B(\mathcal{H})$$

where $\psi_n \circ \phi_n \to \pi$ pointwise in norm.

A separating family

It remains to show that $\{\phi_n\}$ (and hence $\{\rho_{n,k} \circ \phi_n\}$) is separating, which follows from the following theorem.

Theorem (C.,2017)

Any faithful, nondegenerate representation $(\pi, \mathcal{H})$ *of* $C^*_u\langle x : \|x\| \leq 1\rangle$ *on a separable Hilbert space asymptotically factorizes through* $\{\phi_n\}$.

The key is to argue that $\pi(x)$ is the norm limit of a sequence of nilpotent contractions, the proof of which relies heavily on the characterization of the norm closure of nilpotents in $B(\mathcal{H})$ due to Apostol-Foias-Voiculescu (1974).

In one variable

Proposition (von Neumann's Inequality for $*$-polynomials)

Let T be a contractive Hilbert space operator, and q be any noncommutative $*$-polynomial. Then,

$$\|q(T)\| \leq \sup_n \{\|q(a)\| : a \in M_n(\mathbb{C}), \|a\| \leq 1\}.$$

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Andô's inequality for $*$-polynomials?

Theorem (Andô, 1963)

Let T_1 and T_2 be commuting contractions on a Hilbert space, and let $p \in \mathbb{C}[z_1, z_2]$ be a polynomial in two variables. Then

$$\|p(T_1, T_2)\| \leq \sup_{z_1, z_2 \in \overline{\mathbb{D}}} |p(z_1, z_2)|.$$

Andô's inequality for ∗-polynomials?

Theorem (Andô, 1963)

Let T_1 and T_2 be commuting contractions on a Hilbert space, and let $p \in \mathbb{C}[z_1, z_2]$ be a polynomial in two variables. Then

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Can we have an analogous theorem for non-commutative ∗-polynomials in two variables?

Doubly Commuting Contractions

For any noncommutative $*$-polynomial q in two variables, we can try to maximize the norm of $\|q(T_1, T_2)\|$ as T_1 and T_2 range over pairs of commuting contractions.

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For any noncommutative ∗-polynomial q in two variables, we can try to maximize the norm of $\|q(T_1, T_2)\|$ as T_1 and T_2 range over pairs of commuting contractions.

Let's just start with the "nicer" case where T_1 and T_2 range over pairs of doubly commuting contractions, i.e. $T_1T_2 = T_2T_1$ and $T_1T_2^* = T_2^*T_1$.

The question(s)

Question (1)

Given any pair T_1 and T_2 of d.c. contractions on a Hilbert space and any noncommutative $*$-polynomial q in two variables, must the following inequality hold?

$$\|q(T_1, T_2)\| \leq \sup_n \{\|q(a_1, a_2)\| : a_i \in M_n, \|a_i\| \leq 1, a_i \text{ d.c.}\}$$

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Question (2)

Given any pair T_1 and T_2 of d.c. contractions on a Hilbert space and any noncommutative ∗-polynomial q in two variables, must the following inequality hold?

$$\|q(T_1, T_2)\| \leq \sup_n \{\|q(a_1, a_2)\| : a_i \in M_n, \|a_i\| \leq 1, a_i \text{ d.c.}, a_i^n = 0\}$$

"Answers"

Theorem (C., 2017)

The following are equivalent

1. *Question (1) has an affirmative answer.*
2. *Question (2) has an affirmative answer.*
3. *Connes' Embedding Problem has an affirmative answer.*

In other words

Notice that Question (1) asks whether

$$C_u^*\langle x_1, x_2 : \|x_i\| \leq 1 \text{ and } x_1, x_2 \text{ d.c.}\rangle$$

is RFD.

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Notice that Question (1) asks whether

$$C_u^*\langle x_1, x_2 : \|x_i\| \leq 1 \text{ and } x_1, x_2 \text{ d.c.}\rangle$$

is RFD. Question (2) asks whether its separating family of finite-dimensional representations can be chosen so that the generators map to nilpotents.

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Notice that Question (1) asks whether

$$C^*_u\langle x_1, x_2 : \|x_i\| \leq 1 \text{ and } x_1, x_2 \text{ d.c.}\rangle$$

is RFD. Question (2) asks whether its separating family of finite-dimensional representations can be chosen so that the generators map to nilpotents. But

$$C^*_u\langle x_1, x_2 : \|x_i\| \leq 1 \text{ and } x_1, x_2 \text{ d.c.}\rangle \simeq C^*_u\langle x : \|x\| \leq 1\rangle \otimes_{max} C^*_u\langle x : \|x\| \leq 1\rangle.$$

In other words

Theorem (Kirchberg, 1993)

Connes Embedding Problem has an affirmative answer iff $C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2)$ is RFD.

In other words

So, the theorem can be restated as follows.

Theorem (C., 2017)

The following are equivalent

1. $C_u^*\langle x : \|x\| \leq 1\rangle \otimes_{max} C_u^*\langle x : \|x\| \leq 1\rangle$ *is RFD.*
2. $C_u^*\langle x : \|x\| \leq 1\rangle \otimes_{max} C_u^*\langle x : \|x\| \leq 1\rangle$ *is RFD, and its separating family of finite-dimensional representations can be chosen so that the generators map to nilpotents.*
3. $C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2)$ *is RFD.*

Thank you.

Some Recommended Reading

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