Nuclearity and generalized inductive limits

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joint with Wilhelm Winter

WWU Münster

Conference on Operator Algebras and Related Topics
In Memory of Vaughan Jones
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The UVa Department of Mathematics & The Institute for Mathematical Sciences

present

Virginia Mathematics Lectures

Charlottesville, Virginia

Fields Medalist Vaughan F. R. Jones

Vanderbilt University

Knots and Groups

Monday April 6, 4-5 PM in Clark 108

Von Neumann Algebras and Physics

Tuesday April 7, 5-6 PM in Clark 108

Do All Subfactors Arise in Conformal Field Theory?

Wednesday April 8, 4-5 PM in Monroe 130

More Information: www.math.virginia.edu/lectures

Organizers: Andrei Rapinchuk and David Sherman

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Part I: Inductive limits of C^*-algebras

Inductive limits of C^*-algebras

An inductive system of C^*-algebras consists of a sequence $(A_n)_n$ of C^*-algebras together with connecting *-homomorphisms

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For each $k \geq 0$, the quotient map $\prod_n A_n \to {}^{\prod_n A_n}/_{\bigoplus_n A_n}$ induces a *-homomorphism $\rho_k : A_k \to {}^{\prod_n A_n}/_{\bigoplus_n A_n}$ by

$$\rho_k(a) := [\big(\rho_{n,k}(a)\big)_{n>k})], \ \forall \ a \in A_k.$$

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The inductive limit of the system $(A_n, \rho_{m,n})$ is the C^*-algebra

$$A := \overline{\bigcup_{k\geq 0} \rho_k(A_k)} \subset \prod_n A_n / \bigoplus_n A_n.$$

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$$\mathrm{M}_2 \xhookrightarrow{a \mapsto a \oplus a} \mathrm{M}_4 \hookrightarrow \ldots \hookrightarrow \overline{\bigcup_{k=1}^{\infty} \mathrm{M}_{2^k}} =: \mathrm{M}_{2^\infty}$$

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Alternatively, an AF C^*-algebra is one that contains an ascending sequence of finite dimensional subalgebras with norm-dense union. This has an important von Neumann analogue.

AFD von Neumann Algebras

Definition

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There are other ways to approximate operator algebras by finite dimensional ones.

Semi-discrete von Neumann Algebras

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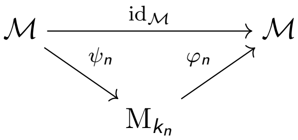

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$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad \mathrm{id}_{\mathcal{M}} \quad} & \mathcal{M} \\ & \searrow^{\psi_n} \quad {}^{\varphi_n}\nearrow & \\ & \mathrm{M}_{k_n} & \end{array}$$

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Theorem (Connes)

Any semi-discrete von Neumann algebra is AFD.

Classification of von Neumann factors

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Corollary (Connes, Murray-von Neumann)

The following von Neumann algebras are all *-isomorphic to $\mathcal{R}$:

- $L(\Gamma)$ *for a countable ICC amenable group* Γ
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The analogous classification of nuclear C^*-algebras is not *nearly* as tidy, and was only recently completed after the work of many hands over many years.

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$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ & {\scriptstyle\psi_n}\searrow \quad \nearrow{\scriptstyle\varphi_n} & \\ & \mathrm{M}_{k_n} & \end{array}$$

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Example

Any AF C^*-algebra is nuclear.

Nuclear and AF C^*-algebras

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- $C(X)$ where X is an infinite totally disconnected compact metrizable space.
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So, a direct analogue to Connes' result is out of the question.

But this is neither unusal nor a deterrent.

Part II: Generalized Inductive Limits

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Asymptotic behavior is what really matters.

Following this philosophy, Blackadar and Kirchberg introduced geneneralized inductive systems of C^*-algebras, where the connecting maps only asymptotically behave like *-homomorphisms. They showed that the limits of such systems form important classes of C^*-algebras.

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Ignoring the full generality of their constructions, we focus on their so-called NF systems.

NF systems

Definition

An NF system consists of a sequence $(F_n)_n$ of finite dimensional C^*-algebras together with asymptotically multiplicative cpc maps

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \dots .$$

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Asymptotically multiplicative means that for any $k \geq 0$, $x, y \in F_k$, and $\varepsilon > 0$, there exists an $M > k$ such that for all $m > n > M$,

$$\|\rho_{m,n}\big(\rho_{n,k}(x)\rho_{n,k}(y)\big) - \rho_{m,k}(x)\rho_{m,k}(y)\| < \varepsilon.$$

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Remark

The naive approach would involve $\|\rho_{m,k}(xy) - \rho_{m,k}(x)\rho_{m,k}(y)\|$*, but this is places too much importance on something happening at the* k^{th} *step and is hence too restrictive.*

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The limit of an NF system is formed the same as before: $\overline{\bigcup_k \rho_k(F_k)} \subset \prod_n F_n / \bigoplus F_n$ where $\rho_k : F_k \to \prod_n F_n / \bigoplus F_n$ are the induced cpc maps.

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Theorem (Blackadar-Kirchberg)

A separable C^*-algebra A is NF iff there exists a sequence of finite dimensional C^*-algebras $(F_n)_n$ and cpc maps $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ so that for all $a, b \in A$,

$$\begin{aligned} \|\varphi_n \circ \psi_n(a) - a\| &\to 0, \\ \big|\|a\| - \|\psi_n(a)\|\big| &\to 0, \text{ and} \\ \|\psi_n(a)\psi_n(b) - \psi_n(ab)\| &\to 0. \end{aligned}$$

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How can we get an inductive limit description of these C^*-algebras?

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To drop quasidiagonality from the inductive limits, we must relax the asymptotically multiplicative assumption in our NF systems

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But without this the inductive limit is only a closed self-adjoint subspace of $\prod_n F_n / \bigoplus F_n$, not an algebra.

We need to relax multiplicativity without losing the C^*-structure.

Part III: Order Zero Maps

Completely positive order zero maps

Definition

A cp map $\psi : A \to B$ between C^*-algebras is called order zero if it is orthogonality preserving:

$$ab = 0 \implies \psi(a)\psi(b) = 0, \ \forall \ a, b \in A_+.$$

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- $\mathbb{R}_+$-weighted characters $\lambda\pi : A \to \mathbb{C}$ on a C^*-algebra A.
- For the function $\mathrm{id} : z \to z$ in $C([0,1])$, the map

  $$M_{\mathrm{id}} : C([0,1]) \to C([0,1]),$$

  given by $M_{\mathrm{id}}(g)(z) = zg(z)$ for $g \in C([0,1])$ is cp order zero.

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  given by $M_{\mathrm{id}}(g)(z) = zg(z)$ for $g \in C([0,1])$ is cp order zero.
- Given a *-homomorphism $\pi : A \to B$ between C^*-algebras and $h \in \pi(A)' \cap B_+$, the map $h\pi(\cdot) : A \to B$ is cp order zero.

A structure theorem for order zero maps

Theorem (Winter-Zacharias)

Every cp order zero map can be written as $h\pi(\cdot) : A \to B$ *for some* **-homomorphism* $\pi : A \to B$ *and* $h \in \pi(A)' \cap B_+$.

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Corollary (Wolff, Winter-Zacharias)

Let A *and* B *be* C^**-algebras with* A *unital. A cp map* $\psi : A \to B$ *is order zero iff*

$$\psi(a)\psi(b) = \psi(1_A)\psi(ab), \ \forall \ a, b \in A.$$

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Remark

Note that if $\psi(1_A) = 1_B$, *then* ψ *is a* $*$*-homomorphism.*

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If $\psi : A \to B$ *is a cp order zero map, then so are all of its matrix amplifications* $\psi^{(r)} : \mathrm{M}_r(A) \to \mathrm{M}_r(B)$*.*

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In other words, an order zero map is completely order zero.

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If $\psi : A \to B$ *is a cp order zero map, then so are all of its matrix amplifications* $\psi^{(r)} : \mathrm{M}_r(A) \to \mathrm{M}_r(B)$*.*

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[1] One can view $(\psi(A), \{\mathrm{M}_r(\psi(A)) \cap \mathrm{M}_r(B)_+\}_r, \psi(1_A))$ as an abstract operator system.

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Theorem (C.-Winter)

Let B be a C^*-algebra, $X \subset B$ a closed self-adjoint subspace, and $e \in B^1_+$ a distinguished element satisfying

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Shorthand

For a closed self-adjoint subspace X of a C^*-algebra B with distinguished element $e \in B^1_+$, we abbreviate the criteria that gave us a C^*-structure on X as follows:

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Whenever (X, e) satisfy (C^*), we can define multiplication $\bullet : X \times X \to X$ and a C^*-norm $\| \cdot \|_\bullet$ on $(X, \bullet)$, and denote the corresponding C^*-algebra with $\mathrm{C}^*_\bullet(X)$.

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If (X, e) arose as the image of an injective cpc order zero map from some unital C^*-algebra $\psi : A \to B$, then $A \cong \mathrm{C}^*_\bullet(X) \cong \mathrm{C}^*_{\mathrm{min}}(X)$.

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How special was e?

If $h \in B^1_+$ is another element so that (X, h) also satisfy (C^*), then the associated C^*-algebras would be unitally *-isomorphic.

Part IV: Generalized NF Systems

Back to generalized inductive limits

Recall that our goal was to relax the "asymptotic multiplicative" requirement from the NF systems:

Definition

An NF system consists of a sequence $(F_n)_n$ of finite dimensional C^*-algebras together with asymptotically multiplicative cpc maps

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \dots$$

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Now we are equipped to overcome that hurdle.

Generalizing generalized inductive limits

Given a sequence $(F_n)_n$ of finite dimensional C^*-algebras together cpc connecting maps

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \ldots,$$

we still have induced cpc maps $\rho_k : F_k \to \prod_n F_n / \bigoplus_n F_n =: F_\infty$, and we can still form the limit

$$X = \overline{\bigcup_k \rho_k(F_k)} \subset F_\infty.$$

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Though X may not be a C^*-algebra, if we can guarantee that there some $e \in (F_\infty)^1_+$ so that (X, e) satisfy (C^*), then it will be completely order isomorphic to the C^*-algebra $\mathrm{C}^*_\bullet(X)$ via the injective cpc order zero map $\mathrm{id}_X : \mathrm{C}^*_\bullet(X) \to X \subset F_\infty$.

Encoding (C^*)

We say a sequence $(F_n)_n$ of finite dimensional C^*-algebras together cpc connecting maps

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This definition is build to ensure that the limit $X = \overline{\bigcup_n \rho_n(F_n)} \subset F_\infty$ together with $e = [(e_n)_n)]$ satisfy (C^*).

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Example

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Example (BK, WZ, Brown-Carrión-White, CW)

Any separable, unital, nuclear C^*-algebra A admits a cpc approximation that gives rise to a generalized NF system.

Generalized NF systems from cpc approximations

Consider a cpc approximation $A \xrightarrow{\psi_n} F_n \xrightarrow{\varphi_n} A$ of a unital nuclear C^*-algebra with asymptotically order zero maps $(\psi_n : A \to F_n)_n$.

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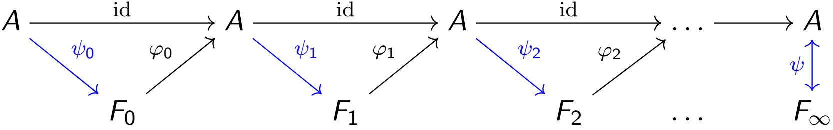

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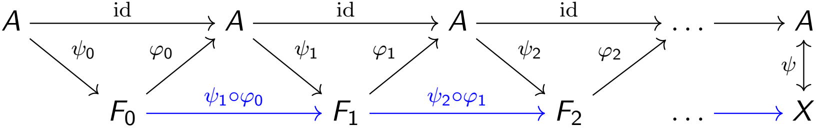

After passing to a subsystem, we can guarantee that $\psi(A) = \varinjlim(F_n, \psi_m \circ ... \circ \varphi_n) =: X$.

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$$\begin{array}{ccccccccccc}
A & & \xrightarrow{\mathrm{id}} & & A & & \xrightarrow{\mathrm{id}} & & A & \xrightarrow{\mathrm{id}} \ \dots \longrightarrow & A \\
& \searrow^{\psi_0} & & \nearrow^{\varphi_0} & & \searrow^{\psi_1} & & \nearrow^{\varphi_1} & & \searrow^{\psi_2} \quad \nearrow^{\varphi_2} & \updownarrow \psi \\
& & F_0 & \xrightarrow{\psi_1 \circ \varphi_0} & & & F_1 & \xrightarrow{\psi_2 \circ \varphi_1} & & F_2 \qquad \dots \longrightarrow & X
\end{array}$$

After passing to a subsystem, we can guarantee that $\psi(A) = \varinjlim(F_n, \psi_m \circ \ldots \circ \varphi_n) =: X$.

The fact that ψ is cpc order zero will imply that (X, e) satisfy (C^*) and moreover that the system is generalized NF.

Limits of generalized NF systems from cpc approximations

Since X is the image of an injective cpc order zero map, we have moreover that $A \cong \mathrm{C}^*_\bullet(X)$.

Theorem (C.-Winter)

Any separable, unital, nuclear C^*-algebra is completely isometrically completely order isomorphic to the limit of a generalized NF system via an order zero map.

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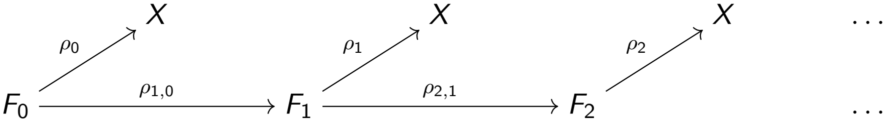
Given any generalized NF system, will the associated C^*-algebra $\mathrm{C}^*_\bullet(X)$ be nuclear?

CPAP for a generalized NF system?



CPAP for a generalized NF system?

$$
\begin{array}{ccccccc}
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 & & \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle \mathrm{id}_X} & \\
 & & X & & X & & X & \cdots \\
 & \nearrow{\scriptstyle \rho_0} & & \nearrow{\scriptstyle \rho_1} & & \nearrow{\scriptstyle \rho_2} & & \\
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\end{array}
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Since id_X^{-1} is cp, so are $\varphi_n := \mathrm{id}_X^{-1} \circ \rho_n$.

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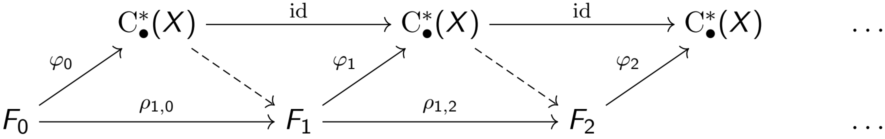

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Can we come up with the downwards maps to get a completely positive approximation?

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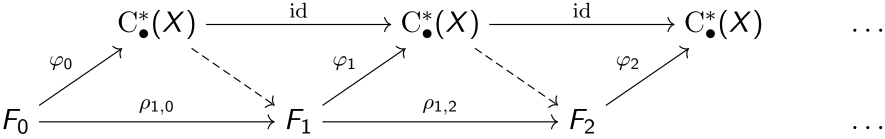

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[Winter] If we assume the upwards maps are decomposable into a direct sum of a bounded number of cpc order zero maps, then yes.

CPAP for a generalized NF system?

$$\begin{array}{ccccccccc}
 & & \mathrm{C}^*_\bullet(X) & \xrightarrow{\mathrm{id}} & & \mathrm{C}^*_\bullet(X) & \xrightarrow{\mathrm{id}} & & \mathrm{C}^*_\bullet(X) & \cdots \\
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Nonetheless, this picture fits perfectly into Ozawa and Sato's "one-way CPAP," which then tells us that $\mathrm{C}^*_\bullet(X)^{**}$ is injective.

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Using again Connes' theorem, we can conclude that $\mathrm{C}^*_\bullet(X)^{**}$ is semi-discrete, and hence that $\mathrm{C}^*_\bullet(X)$ is nuclear.

Nuclear C^*-algebras from limits of generalized NF systems

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The inductive limit X of a generalized NF system is completely order isomorphic to a unital nuclear *C^*-algebra $\mathrm{C}^*_\bullet(X)$ via an completely isometric cp order zero map $\mathrm{id}_X : \mathrm{C}^*_\bullet(X) \to X \subset F_\infty$.*

Removing quasidiagonality

Recall Blackadar and Kirchberg's characterization of NF algebras as the separable nuclear quasidiagonal C^*-algebras:

Theorem (Blackadar-Kirchberg)

The following are equivalent for a separable C^*-algebra A:

1. *A is nuclear and quasidiagonal.*
2. *A is *-isomorphic to an NF algebra.*

Removing quasidiagonality

Recall Blackadar and Kirchberg's characterization of NF algebras as the separable nuclear quasidiagonal C^*-algebras:

Theorem (Blackadar-Kirchberg)

The following are equivalent for a separable C^*-algebra A:

1. *A is nuclear and quasidiagonal.*
2. *A is *-isomorphic to an NF algebra.*

By replacing asymptotic multiplicativity with asymptotic order zero, we can drop "quasidiagonal."

Theorem (C.-Winter)

The following are equivalent for a separable C^*-algebra A:

1. *A is nuclear.*
2. *A is completely isometrically completely order isomorphic to the limit of a generalized NF system via an order zero map.*


Thank you.

Epilogue: Technical Details

Encoding (C^*)

The task is to encode (C^*) into a system

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \dots$$

of cpc maps between finite dimensional C^*-algebras.

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of cpc maps between finite dimensional C^*-algebras.

We want conditions on the system which guarantee that we have an element $e \in (F_\infty)^1_+$ so that the limit X together with e satisfy

$$(\mathrm{C}^*) \begin{cases} 1.\ e \in X' \cap X \\ 2.\ X^2 = eX, \text{ and} \\ 3.\ e \text{ is an order unit for } X. \end{cases}$$

An approximately central order unit

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- asymptotic uniform order unit, which guarantees that $\|\rho_n(x)\| e \geq \rho_n(x)$ for every $n \geq 0$, $x = x^* \in F_n$.

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Let's call such a sequence an asymptotically central order unit.

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We arrange for this by requiring that our system be

- asymptotically order zero with respect to $(e_n)_n$.
  This condition tells us how to build, for any $k \geq 0$ and $x, y \in F_k$, an element $z \in X = \overline{\bigcup_n \rho_n(F_n)}$ so that $ez = \rho_k(x)\rho_k(y)$.

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Remark

Just as with order zero maps, if these maps are asymptotically unital (i.e. $\|e_n - 1_{F_n}\| \to 0$), then the resulting sequence is asymptotically multiplicative, and we land back in the NF setting.

Generalized NF systems

(Working title)

Definition (C.-Winter)

A generalized NF system $(F_n, \rho_{m,n}, e_n)$ consists of a sequence $(F_n)_n$ of finite dimensional C^*-algebras with cpc connecting maps

$$F_0 \xrightarrow{\rho_{1,0}} F_1 \xrightarrow{\rho_{2,1}} F_2 \to \ldots,$$

that are asymptotically order zero with respect to an asymptotically central order unit $(e_n)_n \in \prod_n (F_n)_+^1$.

One Way CPAP

Theorem (Sato, Ozawa)

A C^*-algebra is nuclear iff there exists a net $(\rho_\lambda : F_\lambda \to A)_{\lambda\in\Lambda}$ of cpc maps from finite dimensional C^*-algebras such that the induced cpc map

$$\begin{array}{ccc} \prod_\lambda F_\lambda & \xrightarrow{(\rho_\lambda)_\lambda} & \ell^\infty(\Lambda, A) \\ \downarrow & & \downarrow \\ \prod_\lambda F_\lambda \big/ \bigoplus_\lambda F_\lambda & \xrightarrow{\Phi} & \ell^\infty(\Lambda, A) \big/ c_0(\Lambda, A) \end{array}$$

satisfies $A^1 \subset \Phi\left(\left(\frac{\prod_\lambda F_\lambda}{\bigoplus_\lambda F_\lambda}\right)^1\right)$.

Theorem (C.-Winter)

Let B be a C^*-algebra, $X \subset B$ a self-adjoint subspace, and $e \in X$ a distinguished element satisfying

1. $0 \leq e \leq 1$
2. $e \in X'$
3. $X^2 = eX$, *and*
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$$xy = e(x \bullet y) \ \forall\ x, y \in X$$

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so that $(X, \bullet)$ is a $*$-algebra with unit e. Moreover, there exists a pre-C^*-norm $\| \cdot \|_\bullet$ on $(X, \bullet)$.

We denote the C^*-algebra $(\overline{X}^{\| \cdot \|_\bullet}, \bullet, \| \cdot \|_\bullet)$ by $\mathrm{C}^*_\bullet(X)$.

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defines an associative bilinear map $\bullet : \theta(C) \times \theta(C) \to \theta(C)$, which satisfies

$$\theta(a)\theta(b) = \theta(1_C)\theta(ab) = \theta(1_C)\big(\theta(a) \bullet \theta(b)\big)$$

for all $\theta(a), \theta(b) \in \theta(C)$.

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Without an order zero map, we cannot say exactly what $x \bullet y$ is for $x, y \in X$, but we can still say that it exists.

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Proposition (C.-Winter)

Let B be a C^*-algebra, $X \subset B$ a closed self-adjoint subspace, and $e \in B_+^1$ a distinguished element satisfying

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1. *$x = y \ \Leftrightarrow\ e^n x = e^n y$ for some $n \geq 1$,*
2. *$x = x^* \ \Leftrightarrow\ (ex) = (ex)^*$, and*
3. *$x \geq 0 \ \Leftrightarrow\ ex \geq 0$,*

where the multiplication is in B.

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And moreover this assignment defines an associative bilinear map making $(X, \bullet)$ into a $*$-algebra.

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In general e has an approximate inverse in B, i.e. a sequence $(h_k(e))_{k\in\mathbb{N}}$ in B where $h_k \in C_0((0,1])$ with $th_k(t) \to 1$ pointwise.

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