Local Lifting and Weak Expectations

Lecture 2: Enter tensor products

Kristin Courtney

University of Southern Denmark

2026, Oxford

Recall: (L)LP

Definition

A unital C^*-algebra A has the (local) lifting property (L)LP if any ucp map $\varphi : A \to B/J$ has a (local) ucp lift.

$$
\begin{tikzcd}
 & & B \arrow[d, "\pi"] \\
\mathcal{S} \arrow[urr, dashed, red] & \subset \; A \arrow[ur, dashed, blue] \arrow[r, "\phi"] & B/J
\end{tikzcd}
$$

Theorem (Kirchberg)

For any (countable) free group $\mathbb{F}$, $\mathrm{C}^*(\mathbb{F})$ *has the LLP (LP).*

Proposition

Let A *be a unital* C^**-algebra and* $\mathbb{F}$ *a free group with* $I \lhd \mathrm{C}^*(\mathbb{F})$ *such that* $A \cong \mathrm{C}^*(\mathbb{F})/I$.
Then A *has the LLP iff* $\mathrm{id}_{\mathrm{C}^*(\mathbb{F})/I}$ *locally lifts.*
For A *separable and* $\mathbb{F}$ *countable,* A *has the LP iff* $\mathrm{id}_{\mathrm{C}^*(\mathbb{F})/I}$ *lifts.*

What about $\mathrm{C}^*_\lambda(\Gamma)$?

For Γ countable, discrete hyperlinear, Rădulescu's theorem says

$$\mathrm{C}^*_\lambda(\Gamma) \quad \subset \quad (L\Gamma, \tau_\lambda) \hookrightarrow \mathcal{R}^\omega$$

Proposition (Kirchberg)

A ucp map $\varphi : A \to B/J$ from a separable C^*-algebra A with LLP into the quotient of a (Q)WEP C^*-algebra B is ucp liftable.

Theorem (Connes, Kirchberg)

Let A be a separable unital C^*-algebra with tracial state τ. TFAE:

1. *τ is amenable.*
2. *There exists a trace-preserving embedding $\pi_\tau(A)'' \subset \mathcal{R}^\omega$ such that the *-homomorphism $\pi_\tau : A \to \pi_\tau(A)'' \subseteq \mathcal{R}^\omega$ has a ucp lift $A \to \ell^\infty(\mathcal{R})$.*

Permanence

Theorem

The LLP is preserved by a number of constructions including

- *Extensions of LLP by LLP [Kirchberg]*
- *Extensions of amenable groups by LLP groups [Fournier-Facio–Willett]*
- $\otimes$ *nuclear* C^**-algebras. (LP too) [Kirchberg]*
- *limits of nested unions of LLP (LP too with rwi inclusions) [Kirchberg]*
- *Full amalgamated free products over finite-dimensional* C^**-algebras (LP too) [Boca, Pisier, Ozawa, Shulman–Eilers]*

Example

Some groups with the LP ($\mathrm{C}^*(\Gamma)$ has the LP) include

- $\mathbb{Z}_m^{*k} = \overbrace{\mathbb{Z}_m * \ldots * \mathbb{Z}_m}^{k}$, for $k, m \geq 1$ [Boca]
- $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ [Kirchberg]
- Virtually free-by-cyclic groups [Fournier-Facio–Willett]
- Limit groups [Fournier-Facio–Willett]
- Baumslag-Solitar groups $BS(n, m)$ [Fournier-Facio–Willett]
- Certain one-relator or right-angled Artin groups [Fournier-Facio–Willett]

Non-Examples for $\mathrm{C}^*(\Gamma)$

Theorem (Ozawa, Gromov)

There exists a group without LP.

Theorem (Thom)

Produced a group which is hyperlinear, property (T), and not RF.[1]
Hence no LLP.

Theorem (Ioana–Spaas–Wiersma)

Let Γ *be a discrete group.*

- *If* Γ *contains* $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ *as a subgroup, (including* $SL_n(\mathbb{Z}), n \geq 3$*), then* $\mathrm{C}^*(\Gamma)$ *does not have the LLP.*
- *If* Γ *is countable with property (T) which is either not finitely presented or* $H^2(\Gamma, \mathbb{Z}\Gamma) \neq 0$*, then* Γ *does not have the LP.*

[1] [Kirchberg] For Γ property (T), RF $\equiv \tau_\lambda \in T(\mathrm{C}^*(\Gamma))$ is amenable.

Hard Questions

1. Does $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ have the (L)LP? (Ozawa)
2. Does LLP imply LP for separable C^*-algebras?
3. Are there any property (T) groups with the LLP?
4. For A separable, is LLP equivalent to $\mathrm{Ext}(A)$ being a group[2]?

Theorem (Kirchberg)

A has the LLP iff $Ext(CA)$ is a group with $CA = C_0((0,1], A)$.

Theorem (Enders–Shulman)

The answer to (4) is yes for certain inductive limits of LLP C^*-algebras.

[2] i.e., does every *-homomorphism $A \to B(\mathcal{H})/\mathcal{K}$ have a ccp lift

C^*-tensor products

Definition (Tensor Product Norms)

For C^*-algebras A and B and $x = \sum_{i=1}^{n} a_i \otimes b_i \in A \odot B$ (the algebraic tensor product)

- The max norm:
  $\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi : A \odot B \to B(\mathcal{H}_\pi) \text{ is a }^*\text{-rep }\}$
- The spatial norm:
  $\|x\|_{\min} = \|\pi_A \otimes \pi_B(x)\|_{B(\mathcal{H}_A \otimes \mathcal{H}_B)}$
  where $(\pi_A, \mathcal{H}_A)$ and $(\pi_B, \mathcal{H}_B)$ are any faithful reps of A and B, resp.

The completions are

$$A \otimes_{\max} B = \overline{A \odot B}^{\|\cdot\|_{\max}} \quad \text{and} \quad A \otimes_{\min} B = \overline{A \odot B}^{\|\cdot\|_{\min}}.$$

Key Facts

Theorem (Universal Property of $\otimes_{\max}$)

*For any C^*-algebra C, any *-hom $A \odot B \to C$ extends uniquely to a *-hom $A \otimes_{max} B \to C$.*
*In particular, if $\pi_A : A \to C$ and $\pi_B : B \to C$ are *-homs with commuting ranges, then $\pi_A \times \pi_B$ extends uniquely to $A \otimes_{max} B$.*

Theorem (Takesaki)

For any $x \in A \odot B$ and any C^*-norm $\|\cdot\|_\alpha$ on $A \odot B$,

$$\|x\|_{min} \leq \|x\|_\alpha \leq \|x\|_{max}.$$

*In other words, we have natural surjective *-homomorphisms*

$$A \otimes_{max} B \to A \otimes_\alpha B \to A \otimes_{min} B.$$

Nuclear

Definition

A C^*-algebra A is nuclear if for any C^*-algebra B, $A \odot B$ has a unique C^*-tensor norm.

Theorem (Choi–Effros, Kirchberg)

This is equivalent to id_A approximately factorizing through matrix algebras with ccp maps, i.e., A has the completely positive approximation property.

Example

$M_n(A) \cong M_n \odot A \rightsquigarrow M_n \odot A = M_n \otimes_{\max} A = M_n \otimes_{\min} A$.

Key Facts

Theorem (Restrictions of Tensor Product Maps)

*If $\pi : A \odot B \to B(\mathcal{H})$ is a *-hom, then there exist restrictions*

$$\pi_A : A \to B(\mathcal{H}) \text{ and } \pi_B : B \to B(\mathcal{H})$$

with commuting ranges such that $\pi = \pi_A \times \pi_B$.

Proposition (Functorialty of $\otimes_{\mathrm{min}}$ and $\otimes_{\mathrm{max}}$)

For cp maps $\phi : A \to C$ and $\psi : B \to D$ between C^*-algebras, $\phi \odot \psi : A \odot B \to C \odot D$ extends to cp maps

- $\phi \otimes_{max} \psi : A \otimes_{max} B \to C \otimes_{max} D$ *and*
- $\phi \otimes_{min} \psi : A \otimes_{min} B \to C \otimes_{min} D$.

with $\|\phi \otimes_{max} \psi\| = \|\phi \otimes_{min} \psi\| = \|\phi\|\|\psi\|$.

Example

For $\varphi : A \to B$, $\mathrm{id}_{M_n} \otimes \varphi = \varphi^{(n)}$

Operator Space/ System Tensor Products

Given two C^*-algebras A, B and operator spaces (resp. systems) $X \subseteq A$, $Y \subseteq B$, we define $X \otimes_{\mathrm{min}} Y$ to be the closure of $X \odot Y \subset A \otimes_{\mathrm{min}} B$.

Proposition (Functorialty of $\otimes_{\mathrm{min}}$)

For cb (resp. ucp) maps $\phi : W \to Y$ and $\psi : X \to Z$ between operator spaces (resp. systems), $\phi \odot \psi : W \odot X \to Y \odot Z$ extends to a cb (resp. ucp) map

$$\phi \otimes_{min} \psi : W \otimes_{min} X \to Y \otimes_{min} Z$$

with $\|\phi \otimes_{min} \psi\|_{cb} = \|\phi\|_{cb}\|\psi\|_{cb}$.

Embeddings and Exactness

Proposition

For any C^**-algebras* $A \subseteq B$ *and* C*,*

$$A \otimes_{min} C \subseteq B \otimes_{min} C,$$

but this can fail for $\otimes_{max}$*.*

Proposition

For any C^**-algebra* A *and exact sequence* $0 \to J \to B \to B/J \to 0$*,*

$$0 \to J \otimes_{max} A \to B \otimes_{max} A \to (B/J) \otimes_{max} A \to 0$$

is exact, but this can fail for $\otimes_{min}$*.*

Recall: WEP

Definition

A unital C^*-algebra A has the WEP if any embedding $A \subseteq B$ is relatively weakly injective (rwi), i.e., there exists a ucp map $\psi : B \to A^{**}$, which extends the canonical embedding $A \hookrightarrow A^{**}$.

$$\begin{tikzcd} B \arrow[dr, dashed, "\psi"] & \\ \cup & \\ A \arrow[r, hook] & A^{**} \end{tikzcd}$$

Equivalently:

- A embeds rwi into some/ any $B(\mathcal{H})$.
- For every C^*-algebra B with $A \subseteq B$ and $*$-representation π of A there exists a ccp map $\varphi : B \to \pi(A)''$ such that $\varphi|_A = \pi$.

Lance's Tensor Characterization of rwi

Theorem (Lance)

For C^*-algebras $A \subseteq B$. Then TFAE:

1. *A embeds relatively weakly injectively into B.*
2. *$A \otimes_{max} C \subseteq B \otimes_{max} C$ for any C^*-algebra C.*
3. *$A \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq B \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty)$*
4. *$A \otimes_{max} \mathrm{C}^*(\mathbb{F}) \subseteq B \otimes_{max} \mathrm{C}^*(\mathbb{F})$ for some/every non-abelian free group $\mathbb{F}$.*

Lance's Tensor Characterization of rwi

Theorem (Lance)

For C^*-algebras $A \subseteq B$. Then TFAE:

1. *A embeds relatively weakly injectively into B.*
2. *$A \otimes_{max} C \subseteq B \otimes_{max} C$ for any C^*-algebra C.*
3. *$A \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq B \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty)$*
4. *$A \otimes_{max} \mathrm{C}^*(\mathbb{F}) \subseteq B \otimes_{max} \mathrm{C}^*(\mathbb{F})$ for some/every non-abelian free group $\mathbb{F}$.*

$A \subseteq B$ rwi $\Rightarrow A \otimes_{\max} C \subseteq B \otimes_{\max} C, \forall\ C$

Recall: $A \subseteq B$ rwi iff for any $*$-hom $\pi : A \to B(\mathcal{H})$ there exists a ccp map $\rho : B \to \pi(A)''$ extending π.

$$\begin{tikzcd}
B \otimes_{\max} C & & \pi_A(A)'' \otimes_{\max} \pi_C(C) \arrow[d] \\
A \otimes_{\max} C \arrow[u] \arrow[rr, hook, "\pi", "\pi_A \times \pi_C"'] & & B(\mathcal{H})
\end{tikzcd}$$

Recall: Any *-hom $\sigma : A_1 \odot A_2 \to D$ extends to $A_1 \otimes_{\max} A_2$, e.g. $\sigma_1 \times \sigma_2$ where $\sigma_i : A_i \to D$ have commuting ranges.

$A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq B \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) \ \Rightarrow$ same for any $\mathbb{F}$

Proposition

For any discrete groups $\Lambda \leq \Gamma$, *there is a canonical embedding* $\mathrm{C}^*(\Lambda) \subseteq \mathrm{C}^*(\Gamma)$ *and a conditional expectation* $\mathrm{C}^*(\Gamma) \to \mathrm{C}^*(\Lambda)$. *In particular this holds for* $\mathbb{F}_2 \leq \mathbb{F}_n \leq \mathbb{F}_\infty \leq \mathbb{F}_2$.

Corollary

Let $\mathbb{F}$ *be an arbitrary non-abelian free group, then for any finite dimensional subspace* $D \subseteq \mathrm{C}^*(\mathbb{F})$, $\exists\ \mathrm{C}^*(\mathbb{F}_\infty) \hookrightarrow \mathrm{C}^*(\mathbb{F})$ *with*

$$D \subseteq \mathrm{C}^*(\mathbb{F}_\infty) \subseteq \mathrm{C}^*(\mathbb{F})$$

and a conditional expectation $\mathrm{C}^*(\mathbb{F}) \to \mathrm{C}^*(\mathbb{F}_\infty)$.

Remark

By (1) $\Rightarrow$ (2), $C \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq C \otimes_{max} \mathrm{C}^*(\mathbb{F})$ *for* $C = A, B$.

$A \otimes_{\max} \mathrm{C}^*(\mathbb{F}) \subseteq B \otimes_{\max} \mathrm{C}^*(\mathbb{F})\ \forall \mathbb{F} \Rightarrow A \subseteq B$ rwi

Recall: $A \subseteq B$ rwi means $\exists\ \varphi : B \to \pi_u(A)''$ with $\varphi|_A = \pi_u$.

The multiplicative domain of a ccp map $\phi : C \to D$

$$C_\phi = \{a \in C \mid \phi(ab) = \phi(a)\phi(b), \phi(ba) = \phi(b)\phi(a),\ \forall b \in C\}$$

is the largest C^*-subalgebra of C for which φ is a *-homomorphism.

Lance's Trick

Lance's Tensor Product Trick

Let $A \subseteq B$ and C be C^*-algebras such that

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C.$$

Given representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_C : C \to B(\mathcal{H})$ with commuting ranges, there exists a ucp map $\phi : B \to \pi_C(C)'$ which extends π_A.

Definition (New)

For C^*-algebras $A \subseteq B$, A is rwi in B if for any C^*-algebra C,

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C.$$

Corollary

Suppose $A \subseteq B$ rwi, then for any C^*-algebra C,

$$B \otimes_{max} C = B \otimes_{min} C \implies A \otimes_{max} C = A \otimes_{min} C.$$

(Recall) Lance's Tensor Characterization of rwi

Theorem (Lance)

Let B be a C^*-algebra with C^*-subalgebra A. Then TFAE:

1. *A embeds relatively weakly injectively into B.*
2. *$A \otimes_{max} C \subseteq B \otimes_{max} C$ for any C^*-algebra C.*
3. *$A \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq B \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty)$*
4. *$A \otimes_{max} \mathrm{C}^*(\mathbb{F}) \subseteq B \otimes_{max} \mathrm{C}^*(\mathbb{F})$ for some/every non-abelian free group $\mathbb{F}$.*

Lance's Characterization of WEP

Corollary

Let A be a C^*-algebra. Then TFAE:

1. *A has the WEP.*
2. *For any C^*-algebra B with $A \subseteq B$ and any C^*-algebra C,*

$$A \otimes_{max} C \subseteq B \otimes_{max} C$$

3. *For some embedding $A \subseteq B(\mathcal{H})$*

$$A \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq B(\mathcal{H}) \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty)$$

4. *For any embedding $A \subseteq B(\mathcal{H})$ and any free group $\mathbb{F}$*

$$A \otimes_{max} \mathrm{C}^*(\mathbb{F}) \subseteq B(\mathcal{H}) \otimes_{max} \mathrm{C}^*(\mathbb{F})$$

Example

All nuclear C^*-algebras have WEP.

Theorem (Kirchberg)

For any free group $\mathbb{F}$ and Hilbert space $\mathcal{H}$,

$$\mathrm{C}^*(\mathbb{F}) \otimes_{min} B(\mathcal{H}) = \mathrm{C}^*(\mathbb{F}) \otimes_{max} B(\mathcal{H}).$$

We follow Pisier's proof.

Idea: Want $\mathrm{id}_{\mathrm{C}^*(\mathbb{F}_n)\odot B(\ell^2)}$ to extent to a *-homomorphism $\mathrm{C}^*(\mathbb{F}_n) \otimes_{\mathrm{min}} B(\ell^2) \to \mathrm{C}^*(\mathbb{F}_n) \otimes_{\mathrm{max}} B(\ell^2)$.

Lemma (Pisier's Linearization Trick)

Let A, B be unital C^*-algebras, $X \subseteq A$ a unital subspace, and $\varphi : X \to B$ a unital cc map. Suppose there exists $\{a_i\}_{i \in I} \subseteq X$ such that

1. $\|a_i\| \leq 1$ *for all* $i \in I$,
2. $\varphi(a_i)$ *is a unitary for all* $i \in I$, *and*
3. $\mathrm{C}^*(X) = \mathrm{C}^*(\{a_i\}_{i \in I}) \subseteq A$.

Then φ uniquely extends to a $*$-homomorphism $\mathrm{C}^*(X) \to B$.

Theorem (Arveson–Wittstock)

Let A be a untial C^*-algebra with unital subspace X and $\varphi : X \to B(\mathcal{H})$ a unital cc map. Then there exists a ucp map $\tilde{\varphi} : A \to B(\mathcal{H})$ extending φ.

Proposition (Corollary to Stinespring)

The multiplicative domain of a ccp map $\varphi : A \to B$ between C^*-algebras is

$$A_\varphi = \{a \in A \mid \varphi(a^*a) = \varphi(a)^*\varphi(a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a)^*\}.$$

Corollary

For $\varphi : A \to B$ ccp and $a \in A$ with $\|a\| \leq 1$, if $\varphi(a)$ is a unitary, then $a \in A_\varphi$.

Lemma

Let $z = \sum_{k=0}^{n} U_k \otimes x_k \in X_n \otimes_{min} B(\ell^2)$ *with* $\|z\|_{min} = 1$*.*
Then there exist a_k *and* b_k *in* $B(\ell^2)$ *for* $k = 0, ..., n$ *such that*

1. $a_k^* b_k = x_k$ *for each* k *and*
2. $\displaystyle\sum_{k=0}^{n} a_k^* a_k = 1 = \sum_{k=0}^{n} b_k^* b_k$*.*

Proof Idea

For operator spaces X, Y we have

$$\begin{array}{ccc} X^* \otimes_{\min} Y & \xrightarrow[\cong]{\text{completely isometric}} & \mathrm{CB}(X, Y) \\ \sum \varphi_k \otimes y_k & & \left[x \mapsto \sum \varphi_k(x) y_k\right] \end{array}$$

$X_n = \mathrm{span}\{U_0, \dots, U_n\} \cong \ell^1_{n+1} \cong (\ell^\infty_{n+1})^*$ as operator spaces, so

$$\begin{array}{ccc} X_n \otimes_{\min} B(\ell^2) & \xrightarrow{\cong} & \mathrm{CB}(\ell^\infty_{n+1}, B(\ell^2)) \\ z = \sum U_k \otimes x_k & & T_z((\alpha_k)_{k=0}^\infty) = \sum \alpha_k x_k \end{array}$$

Theorem (Haagerup, Paulsen, Wittstock Stinespring Theorem)

Let $X \subseteq A$ be an operator subspace of a C^*-algebra and $T : X \to B(\mathcal{H})$ a cc map. Then there exists a Hilbert space $\hat{\mathcal{H}}$, a unital *-homomorphism $\pi : A \to B(\hat{\mathcal{H}})$, and isometries $V, W : \mathcal{H} \to \hat{\mathcal{H}}$ such that $T(x) = V^* \pi(x) W$ for every $x \in X$.

Lemma

Let $z = \sum_{k=0}^{n} U_k \otimes x_k \in X_n \otimes_{min} B(\ell^2)$ *with* $\|z\|_{min} = 1$*.*
Then there exist a_k *and* b_k *in* $B(\ell^2)$ *for* $k = 0, ..., n$ *such that*

1. $a_k^* b_k = x_k$ *for each* k *and*
2. $\displaystyle\sum_{k=0}^{n} a_k^* a_k = 1 = \sum_{k=0}^{n} b_k^* b_k.$

Now for such a z,

$$1 = \left\| \sum_{k=0}^{n-1} U_k \otimes x_k \right\|_{\max} = \left\| \sum_{k=0}^{n-1} (1 \otimes a_k)^* (U_k \otimes b_k) \right\|_{\max}$$

$$\leq \left\| \sum_{k=0}^{n-1} (1 \otimes a_k)^* (1 \otimes a_k) \right\|_{\max}^{1/2} \left\| \sum_{k=0}^{n-1} (U_k \otimes b_k^*)(U_k \otimes b_k) \right\|_{\max}^{1/2} = 1.$$

Theorem (Kirchberg)

For any C^*-algebra A,

$$A \text{ has the WEP} \iff \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{max} A = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{min} A.$$

The same holds if $\mathbb{F}_\infty$ is replaced by any non-abelian free group.

Recall: A has WEP iff for some embedding $A \subseteq B(\mathcal{H})$

$$A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$$