Local Lifting and Weak Expectations

Lecture 3: Everything, Everywhere, All at Once

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Where we left off...

Theorem (Kirchberg)

For any free group $\mathbb{F}$ and Hilbert space $\mathcal{H}$,

$$\mathrm{C}^*(\mathbb{F}) \otimes_{min} B(\mathcal{H}) = \mathrm{C}^*(\mathbb{F}) \otimes_{max} B(\mathcal{H}).$$

Big Ideas

Restrict to ℓ^2 and $\mathbb{F}_n, n < \infty$ with generators $U_0 = 1, U_1, ..., U_n$.

Want: $\mathrm{id}_{\mathrm{C}^*(\mathbb{F}_n)\odot B(\ell^2)}$ to extent to a *-homomorphism $\mathrm{C}^*(\mathbb{F}_n) \otimes_{\mathsf{min}} B(\ell^2) \to \mathrm{C}^*(\mathbb{F}_n) \otimes_{\mathsf{max}} B(\ell^2)$.

Set $X_n = \mathrm{span}\{U_0, ..., U_n\}$.

By Pisier's Linearization Trick, it suffices to extend $\mathrm{id}_{X_n \odot B(\ell^2)}$ to a cc map

$$X_n \otimes_{\mathsf{min}} B(\ell^2) \to \mathrm{C}^*(\mathbb{F}_n) \otimes_{\mathsf{max}} B(\ell^2).$$

The main work is showing $\mathrm{id}_{X_n \odot B(\ell^2)}$ is contractive.

Key Tool

For operator spaces X, Y we have

$$\begin{array}{ccc} X^* \otimes_{\min} Y & \xrightarrow{\text{isometric}} & \mathrm{CB}(X, Y) \\ \sum \varphi_k \otimes y_k & & \left[x \mapsto \sum \varphi_k(x) y_k\right] \end{array}$$

$X_n = \mathrm{span}\{U_0, \ldots, U_n\} \cong \ell^1_{n+1} \cong (\ell^\infty_{n+1})^*$ as operator spaces, so

$$\begin{array}{ccc} X_n \otimes_{\min} B(\ell^2) & \xrightarrow[\cong]{\text{isometric}} & \mathrm{CB}(\ell^\infty_{n+1}, B(\ell^2)) \\ z = \sum U_k \otimes x_k & & T_z((\alpha_k)_{k=0}^\infty) = \sum \alpha_k x_k \end{array}$$

Kirchberg's Tensor Characterization of WEP/LLP

Theorem (Kirchberg)

Let A and B be C^*-algebras. Then

1. *A has the LLP* $\iff A \otimes_{max} B(\ell^2) = A \otimes_{min} B(\ell^2)$,
2. *B has the WEP* $\iff \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{max} B = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{min} B$, *and*
3. *A has the LLP and B has the WEP* $\Rightarrow A \otimes_{max} B = A \otimes_{min} B$.

One can replace ℓ^2 with any infinite-dimensional Hilbert space and $\mathbb{F}_\infty$ with any non-abelian free group.

Proof of (2)

Recall: A has WEP iff for some embedding $A \subseteq B(\mathcal{H})$

$$A \otimes_{\mathsf{max}} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq B(\mathcal{H}) \otimes_{\mathsf{max}} \mathrm{C}^*(\mathbb{F}_\infty)$$

Effros-Haagerup Lifting Theorem

For a C^*-algebra B and $J \lhd B$, TFAE

1. The following sequence is exact for any C^*-algebra C

$$0 \to C \otimes_{\mathsf{min}} J \to C \otimes_{\mathsf{min}} B \to C \otimes_{\mathsf{min}} B/J \to 0.$$

2. The following sequence is exact for some/any ∞-dim $\mathcal{H}$

$$0 \to B(\mathcal{H}) \otimes_{\mathsf{min}} J \to B(\mathcal{H}) \otimes_{\mathsf{min}} B \to B(\mathcal{H}) \otimes_{\mathsf{min}} B/J \to 0.$$

3. $\mathrm{id}_{B/J}$ is locally liftable.

Theorem (Kirchberg)

For a unital C^*-algebra A, TFAE

1. *A has the LLP.*
2. *$A \otimes_{min} B(\mathcal{H}) = A \otimes_{max} B(\mathcal{H})$ for some/any ∞-dim $\mathcal{H}$.*

Let $\mathrm{C}^*(\mathbb{F}) \triangleright J$ such that $A \cong \mathrm{C}^*(\mathbb{F})/J$ and $\mathcal{H}$ a Hilbert space.

Lemma

The following sequence is exact

$$0 \to B(\mathcal{H}) \otimes_{min} J \to B(\mathcal{H}) \otimes_{min} \mathrm{C}^*(\mathbb{F}) \to B(\mathcal{H}) \otimes_{max} A \to 0.$$

Recall: A has the LLP iff $\mathrm{id}_{\mathrm{C}^*(\mathbb{F})/J}$ locally lifts.

Theorem (Kirchberg's Tensor Characterization of WEP/LLP)

Let A and B be C^*-algebras. Then

1. *A has the LLP $\iff A \otimes_{max} B(\ell^2) = A \otimes_{min} B(\ell^2)$,*
2. *B has the WEP $\iff \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{max} B = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{min} B$, and*
3. *A has the LLP and B has the WEP $\Rightarrow A \otimes_{max} B = A \otimes_{min} B$.*

One can replace ℓ^2 with any infinite-dimensional Hilbert space and $\mathbb{F}_\infty$ with any non-abelian free group.

Theorem (Junge–Pisier)

If $\mathcal{H}$ is infinite-dimensional, then $B(\mathcal{H})$ does not have the LLP.

Corollary

If $A \subseteq B$ rwi, then A has the LLP, resp. WEP, if B does.

Recall: If $A \subseteq B$ rwi, then for any C^*-algebra C,
$B \otimes_{\max} C = B \otimes_{\min} C \Longrightarrow A \otimes_{\max} C = A \otimes_{\min} C$.

Corollary

QWEP also passes to rwi subalgebras.

Corollary

If A^{**} has LLP/(Q)WEP, then so does A.

Kirchberg's Conjectures

Between QWEP and CEP

Connes' Embedding Problem

In his 1976 article, *Classification of injective factors. Cases* II_1, II_∞, III_λ, $\lambda \neq 1$, Connes says the following in the middle of the proof of Theorem 5.1:

> *...We now construct an approximate embedding of N in* $\mathcal{R}$*. Apparently such an imbedding ought to exist for all* II_1*-factors because it does for the regular representation of free groups...*

This is now know as Connes' Embedding Problem:
Every II_1*-factor with separable predual embeds into some ultrapower* $\mathcal{R}^\omega$ *of the hyperfinite* II_1*-factor* $\mathcal{R}$*.*

In 1993, Kirchberg proved this is equivalent to a number of things.

Kirchberg's Conjectures

Theorem (Kirchberg)

The following are equivalent:

1. $\mathrm{C}^*(\mathbb{F}_\infty) \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty) = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{min} \mathrm{C}^*(\mathbb{F}_\infty)$.
2. $\mathrm{C}^*(\mathbb{F}_\infty)$ *has the WEP.*
3. $\mathrm{C}^*(\mathbb{F})$ *has WEP for every free group.*
4. *All* C^**-algebras are QWEP.*
5. *LLP* $\Rightarrow$ *WEP.*
6. $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ *is RFD.*
7. *Connes' Embedding Problem has an affirmative answer.*

Reference

Theorem (Kirchberg's Tensor Characterization of WEP/LLP)

Let A and B be C^*-algebras. Then

I. *A has the LLP $\iff A \otimes_{max} B(\ell^2) = A \otimes_{min} B(\ell^2)$,*

II. *B has the WEP $\iff \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{max} B = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{min} B$, and*

III. *A has the LLP and B has the WEP $\Rightarrow A \otimes_{max} B = A \otimes_{min} B$.*

One can replace ℓ^2 with any infinite-dimensional Hilbert space and $\mathbb{F}_\infty$ with any non-abelian free group.

Kirchberg's Conjectures

Theorem (Kirchberg)

The following are equivalent:

1. $\mathrm{C}^*(\mathbb{F}_\infty) \otimes_{max} \mathrm{C}^*(\mathbb{F}_\infty) = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{min} \mathrm{C}^*(\mathbb{F}_\infty)$.
2. $\mathrm{C}^*(\mathbb{F}_\infty)$ *has the WEP.*
3. $\mathrm{C}^*(\mathbb{F})$ *has WEP for every free group.*
4. *All* C^**-algebras are QWEP.*
5. *LLP* $\Rightarrow$ *WEP.*
6. $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ *is RFD.*
7. *Connes' Embedding Problem has an affirmative answer.*

RFD

Definition

A C^*-algebra A is residually finite dimensional (RFD) if it admits a separating family of finite-dimensional representations, i.e.,

$$A \hookrightarrow \prod M_{n_k}.$$

Theorem (Choi)

$\mathrm{C}^*(\mathbb{F}_2)$ is RFD.

Proposition

For discrete groups Γ and Λ, $\mathrm{C}^*(\Gamma \times \Lambda) \cong \mathrm{C}^*(\Gamma) \otimes_{max} \mathrm{C}^*(\Lambda)$.

Proposition

If A and B are RFD C^*-algebras, then $A \otimes_{min} B = A \otimes_{max} B$ iff $A \otimes_{max} B$ is RFD.

All C^*-algebras are QWEP $\Rightarrow$ [LLP$\Rightarrow$WEP]

Effros-Haagerup Lifting Theorem

For a C^*-algebra B and $J \lhd B$, TFAE

1. The following sequence is exact for any C^*-algebra C

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2. The following sequence is exact for some/any ∞-dim $\mathcal{H}$

$$0 \to B(\mathcal{H}) \otimes_{\mathsf{min}} J \to B(\mathcal{H}) \otimes_{\mathsf{min}} B \to B(\mathcal{H}) \otimes_{\mathsf{min}} B/J \to 0.$$

3. $\mathrm{id}_{B/J}$ is locally liftable.

$\mathrm{C}^*(\mathbb{F}_\infty)$ WEP $\Rightarrow$ CEP

CEP: Every II_1-factor (M, τ) with separable predual embeds into some ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1-factor $\mathcal{R}$.

Amenable Traces

Theorem (Connes, Kirchberg)

Let A be a separable unital C^*-algebra with a tracial state τ. TFAE:

1. *τ is amenable.*
2. *For some/any faithful representation $A \to B(\mathcal{H})$, there exists a ucp map $\Psi : B(\mathcal{H}) \to \pi_\tau(A)''$ extending π_τ.*
3. *There exists a trace-preserving embedding $\pi_\tau(A)'' \subset \mathcal{R}^\omega$ such that the *-homomorphism $\pi_\tau : A \to \pi_\tau(A)'' \subseteq \mathcal{R}^\omega$ has a ucp lift $A \to \ell^\infty(\mathcal{R})$.*

CEP $\Rightarrow$ all C^*-algebras are QWEP

Assume every type II_1-factor with separable predual embeds into some ultrapower of the hyperfinite II_1-factor $\mathcal{R}$

$$\mathcal{R}^\omega = \frac{\ell^\infty(\mathcal{R})}{\{(x_n) : \lim_{n\to\omega} \|x_n\|_{2,\tau} = 0\}}.$$

Recall: QWEP passes to rwi subalgebras.

Proposition

For a tracial von Neumann algebra N with von Neumann subalgebra $M \subset N$, there is a conditional expectation $N \to M$. In particular, $M \subseteq N$ rwi.

Theorem (Popa)

Any tracial von Neumann algebra (M, τ) embeds in a trace preserving way into a II_1-factor, which is separably acting provided that M is.

Proposition

A von Neumann algebra M is semifinite (no type III summand) iff $M = (\bigcup M_i)''$ for an increasing net of tracial von Neumann algebras.

Proposition (Kirchberg)

Let $\{A_i\}_{i \in \mathcal{I}}$ be an increasing net of C^*-subalgebras in $B(\mathcal{H})$. If A_i is (Q)WEP for every $i \in \mathcal{I}$, then so is $(\bigcup A_i)''$.

Takesaki's Modular Theory

As a corollary to Takesaki's Duality Theorem and his Structure Theory for Type III von Neumann algebras, we have the following:

Theorem (Takesaki)

For any von Neumann algebra M, the crossed product $M \rtimes_{\alpha} \mathbb{R}$, where α is the modular action, is semifinite.

In particular, M embeds into a semifinite von Neumann algebra N so that there exists a conditional expectation $N \to M$.

Epilogue: Current goings on

- Tsirelson's Problem
- Very Flexible Stability
- Smith–Ward Problem

Epilogue: Tsirelson's Problem

Exercise

In Kirchberg's conjectures, any discrete group Γ *with the LP and* $\mathbb{F}_2 \leq \Gamma$ *(and RFD for good measure) can replace* $\mathbb{F}_\infty$*.*

In particular, CEP is true iff for any/all $m, n \geq 2$, $(m, n) \neq (2, 2)$

$$\mathrm{C}^*(\mathbb{Z}_m^{*n}) \otimes_{\max} \mathrm{C}^*(\mathbb{Z}_m^{*n}) = \mathrm{C}^*(\mathbb{Z}_m^{*n}) \otimes_{\min} \mathrm{C}^*(\mathbb{Z}_m^{*n}).$$

Let's re-write these with PVMs: For $m, n \geq 2$

$$\mathrm{C}^*(\mathbb{Z}_m^{*n}) = \mathrm{C}^*_u\langle u_1, ..., u_n \mid 1 = u_x^* u_x = u_x u_x^* = u_k^m, 1 \leq x \leq n\rangle$$

$$= \mathrm{C}^*_u\langle E_{x,a} \mid E_{x,a} = E_{x,a}^2 = E_{x,a}^*, \sum_{a \in R} E_{x,a} = 1, \substack{1 \leq x \leq n \\ 1 \leq a \leq n}\rangle$$

Correlation Sets and Tsirelson's Problem

Theorem (Fritz (also Junge, et al.))

$$C_{qs}(m,n) = \{\{\rho(E_{x,a} \otimes E_{y,b})\}_{1 \le x,y \le n}^{1 \le a,b \le m} \mid \rho \in S(\mathrm{C}^*(\mathbb{Z}_m^{*n}) \otimes_{min} \mathrm{C}^*(\mathbb{Z}_m^{*n}))\}$$

$$C_{qc}(m,n) = \{\{\rho(E_{x,a} \otimes E_{y,b})\}_{1 \le x,y \le n}^{1 \le a,b \le m} \mid \rho \in S\big(\mathrm{C}^*(\mathbb{Z}_m^{*n}) \otimes_{max} \mathrm{C}^*(\mathbb{Z}_m^{*n})\big)\}$$

Tsirelson's Problem (modern interpretation)

$C_{qs}(m,n) = C_{qc}(m,n)$ for all m,n.

$\rightsquigarrow$ CEP $\Rightarrow$ Tsirelson's Problem.

Theorem (Ozawa)

The converse holds.

Theorem (Ji–Natarajan–Vidick–Wright–Yuen)

$C_{qs}(m,n) \neq C_{qc}(m,n)$ *for some* $m,n >> 2$.

Epilogue: Stability

Classic: If two matrices almost commute, are they close to commuting matrices?

Let $\|\cdot\|_\alpha$ is denote the Hilbert-Schmidt $\|\cdot\|_{2,\mathrm{tr}_{k_n}}$ or operator norm.

Definition

For countable discrete Γ, an approximate morphism is a sequence of unital maps $\phi_n : \Gamma \to \mathcal{U}_{k_n}$ such that for all $g, h \in G$

$$\|\varphi_n(gh) - \varphi_n(g)\varphi_n(h)\|_\alpha \to 0.$$

Definition

Γ is Hilbert-Schmidt stable, resp. operator norm, stable if for any approximate morphism, there is a sequence of morphisms $\psi_n : \Gamma \to \mathcal{U}_{k_n}$ such that for all $g \in G$

$$\|\varphi_n(g) - \psi_n(g)\|_\alpha \to 0.$$

Stability as Lifting

Approximate representations $\varphi_n \colon \Gamma \to \mathcal{U}(k_n)$ $\longleftrightarrow$ Unital $*$-homomorphisms $\Phi : C^*(\Gamma) \to \dfrac{\prod_n M_{k_n}}{\{\|x_n\|_\alpha \to 0 \text{ (along } \omega)\}}$

Stability says Φ lifts to a *-homomorphism $\Psi : \mathrm{C}^*(\Gamma) \to \prod_n M_{k_n}$.

Remark

$\exists$ *various weakenings of stability including (very) flexible stability.*

Theorem (Eckhart–Shulman '23, Fournier-Facio–Willett '26)

Any unital C^*-algebra which is RFD and has LLP is both HS and operator norm very-flexibly stable.

The Smith–Ward Problem

Definition

The matricial range of $T \in B(\mathcal{H})$ is

$$W(T) = \cup_n \{\varphi(T) \mid \varphi : \mathrm{C}^*(T) \to M_n \text{ ucp}\}.$$

The essential matricial range is $W_e(T) = W(q(T))$ where $q : B(\mathcal{H}) \to B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the quotient map.

Smith–Ward Problem (1980)

For all $T \in B(\mathcal{H})$, there exists $K \in \mathcal{K}$ with $W_e(T) = W(T + K)$.

Theorem (Kavruk)

The Smith–Ward Problem has a positive solution iff every 3-dimensional operator system has the LP.

Resolution to the Smith–Ward Problem

This year, building heavily on Samuel Harris' 4-dimensional non-example, Marcel Scherer gave a three-dimensional operator system $\mathcal{S} \subseteq M_4(\mathrm{C}^*_\lambda(\mathbb{F}_2))$ without LP.

Smith–Ward problem has a negative solution.

Thanks

for coming along

Correlations

- In quantum physics, a measurement with m possible outcomes can be modeled with a PVM $\{E_1, ..., E_m\}$.
- Performing n measurements (each with m outcomes) corresponds to n sets of PVMs $\{\{E_{x,a}\}_{1\leq a\leq m}\}_{1\leq x\leq n}$
- The probability of obtaining an outcome a when measuring property x on a quantum system in the state $\varphi \in S(B(\mathcal{H}))$ is

$$p(a \mid x) = \varphi(E_{x,a}).$$

- Now we take two independent labs, A and B, each conducting n measurements, each with m possible outcomes.

Definition

For $m, n \geq 2$ a correlation is a tuple

$$p = \{p(a, b \mid x, y)\}_{1\leq x,y\leq n}^{1\leq a,b\leq m} \in [0, 1]^{m^2 n^2}$$

such that $\sum_{1\leq a,b\leq m} p(a, b \mid x, y) = 1$.

Correlations

For $m, n \geq 2$, we call a correlation

$$p = \{p(a, b \mid x, y)\}_{1 \leq x,y \leq n}^{1 \leq a,b \leq m} \in [0, 1]^{m^2 n^2}$$

- quantum (spatial) correlation if there exist finite-dimensional (arbitrary) Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, unit vector $\xi \in \mathcal{H}_B \otimes \mathcal{H}_B$, and PVMs $\{E_{x,a}\} \subseteq B(\mathcal{H}_A), \{F_{y,b}\} \subseteq B(\mathcal{H}_B)$ so that

$$p(a, b \mid x, y) = \langle (E_{x,a} \otimes F_{y,b})\xi, \xi \rangle.$$

- quantum commuting correlation if there exist Hilbert space $\mathcal{H}$, unit vector $\xi \in \mathcal{H}$ and PVMs $\{E_{x,a}\} \in \{F_{y,b}\}' \subseteq B(\mathcal{H})$ so that

$$p(a, b \mid x, y) = \langle E_{x,a} F_{y,b} \xi, \xi \rangle.$$

$C_q(m, n) = \{$all possible quantum correlations for (m,n)$\}$

$C_{qs}(m, n) = \{$all possible quantum spatial correlations for (m,n)$\}$

$C_{qc}(m, n) = \{$all possible quantum commuting correlations for (m,n)$\}$

Very Flexibly Stable

Definition (Becker–Lubotzky '20)

The group is very flexibly HS, resp. operator norm, stable if for any approximate representation $\{\varphi_n : \Gamma \to \mathcal{U}_{k_n}\}_n$ there is a sequence representations $\{\pi_n : \Gamma \to \mathcal{U}(\mathbb{C}^{d_n})\}_n$ and isometric inclusions $v_n : \mathbb{C}^{k_n} \to \mathbb{C}^{d_n}$ so that

$$\|\varphi_n(g) - v_n^* \pi_n(g) v_n\|_\alpha \to 0$$

where α is the HS or operator norm.
Flexibly stable means $\frac{k_n}{d_n} \to 1$.

Theorem (Eckhart–Shulman '23, Fournier-Facio–Willett '26)

Any unital C^*-algebra which is RFD and has LLP is both HS and operator norm very-flexibly stable.

Close to commuting

Theorem (Voiculescu)

There exists sequences of unitary matrices $(U_n), (V_n)_n$ *such that* $\|[U_n, V_n]\| \to 0$ *but their distance from the set of commuting unitary pairs remains bounded away from zero, i.e.,* $\mathbb{Z}^2$ *is not operator norm stable.*

Corollary (Bekka)

Amenable RF groups are very flexibly stable.

Theorem (Lubotzky–Salomon '26)

$\mathbb{Z}^2$ *is flexibly stable.*

Theorem (Kirchberg)

For any free group $\mathbb{F}$ and Hilbert space $\mathcal{H}$,

$$\mathrm{C}^*(\mathbb{F}) \otimes_{min} B(\mathcal{H}) = \mathrm{C}^*(\mathbb{F}) \otimes_{max} B(\mathcal{H}).$$

Restrict to ℓ^2 and $\mathbb{F}_n, n < \infty$ with generators $U_0 = 1, U_1, ..., U_n$.

Idea: Want $\mathrm{id}_{\mathrm{C}^*(\mathbb{F}_n) \odot B(\ell^2)}$ to extent to a *-homomorphism $\mathrm{C}^*(\mathbb{F}_n) \otimes_{\mathrm{min}} B(\ell^2) \to \mathrm{C}^*(\mathbb{F}_n) \otimes_{\mathrm{max}} B(\ell^2)$.

Set $X_n = \mathrm{span}\{U_0, ..., U_n\}$. By Pisier's Linearization Trick, it suffices to extend $\mathrm{id}_{X_n \odot B(\ell^2)}$ to a cc map

$$X_n \otimes_{\mathrm{min}} B(\ell^2) \to \mathrm{C}^*(\mathbb{F}_n) \otimes_{\mathrm{max}} B(\ell^2).$$

We show $\mathrm{id}_{X_n \odot B(\ell^2)}$ is contractive, i.e.,
for $z = \sum U_k \otimes x_k \in X_n \odot B(\ell^2)$, $\|z\|_{\mathrm{min}} = 1 \implies \|z\|_{\mathrm{max}} = 1$.

Lemma

Let $z = \sum_{k=0}^{n} U_k \otimes x_k \in X_n \otimes_{min} B(\ell^2)$ *with* $\|z\|_{min} = 1$*.*
Then there exist a_k *and* b_k *in* $B(\ell^2)$ *for* $k = 0, ..., n$ *such that*

1. $a_k^* b_k = x_k$ *for each* k *and*
2. $\displaystyle\sum_{k=0}^{n} a_k^* a_k = 1 = \sum_{k=0}^{n} b_k^* b_k.$

Now for such a z,

$$
\begin{aligned}
1 = \left\| \sum_{k=0}^{n-1} U_k \otimes x_k \right\|_{\max} &= \left\| \sum_{k=0}^{n-1} (1 \otimes a_k)^* (U_k \otimes b_k) \right\|_{\max} \\
&\leq \left\| \sum_{k=0}^{n-1} (1 \otimes a_k)^* (1 \otimes a_k) \right\|_{\max}^{1/2} \left\| \sum_{k=0}^{n-1} (U_k \otimes b_k^*)(U_k \otimes b_k) \right\|_{\max}^{1/2} = 1.
\end{aligned}
$$

Proof Idea for Lemma

For operator spaces X, Y we have

$$X^* \otimes_{\min} Y \xrightarrow{\text{isometric}} \mathrm{CB}(X, Y)$$
$$\sum \varphi_k \otimes y_k \qquad \left[x \mapsto \sum \varphi_k(x) y_k\right]$$

$X_n = \operatorname{span}\{U_0, \ldots, U_n\} \cong \ell^1_{n+1} \cong (\ell^\infty_{n+1})^*$ as operator spaces, so

$$X_n \otimes_{\min} B(\ell^2) \xrightarrow{\cong} \mathrm{CB}(\ell^\infty_{n+1}, B(\ell^2))$$
$$z = \sum U_k \otimes x_k \qquad T_z((\alpha_k)_{k=0}^\infty) = \sum \alpha_k x_k$$

Theorem (Haagerup, Paulsen, Wittstock Stinespring Theorem)

Let $X \subseteq A$ be an operator subspace of a C^*-algebra and $T : X \to B(\mathcal{H})$ a cc map. Then there exists a Hilbert space $\hat{\mathcal{H}}$, a unital *-homomorphism $\pi : A \to B(\hat{\mathcal{H}})$, and isometries $V, W : \mathcal{H} \to \hat{\mathcal{H}}$ such that $T(x) = V^* \pi(x) W$ for every $x \in X$.