Local Lifting and Weak Expectations

Lecture 1: A first look at LLP and WEP

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Completely *** Maps

Definition

An operator space is a closed subspace of a C^*-algebra. An operator system is a unital self-adjoint subspace of a C^*-algebra.

A linear map $\varphi : A \to B$ between C^*-algebras is

- *positive* if $a \geq 0 \Rightarrow \varphi(a) \geq 0 \ \forall a \in A$.
- *completely positive* (cp) if $\varphi^{(n)} : M_n(A) \to M_n(B)$ given by

  $$\varphi^{(n)}([a_{ij}]_{i,j=1}^n) = [\varphi(a_{ij})]_{i,j=1}^n$$

  is positive for all $n \in \mathbb{N}$.
- *completely bounded* (cb) if $\|\varphi\|_{cb} = \sup_n \|\varphi^{(n)}\| < \infty$
- *completely contractive* (cc) if $\|\varphi\|_{cb} \leq 1$

We write *ccp* for cc+cp and *ucp* for unital + cp.

little facts

For $\varphi : A \to B$ linear

- For A unital: cp $\Rightarrow \|\varphi\|_{cb} = \|\varphi(1)\|$ and ucp $\iff$ ucc.
- φ positive $\Rightarrow \varphi(x^*) = \varphi(x)^*$
- Schwarz Inequality: if $\varphi : A \to B$ cp, then

$$\varphi(a^* a) \geq \varphi(a)^* \varphi(a)$$

Big Facts

Theorem (Stinespring's Dilation Theorem)

Let A be a unital C^*-algebra and $\varphi : A \to B(\mathcal{H})$ a ucp map. Then there exists a Hilbert space $\hat{\mathcal{H}} \supset \mathcal{H}$ with orthogonal projection $P_{\mathcal{H}} : \hat{\mathcal{H}} \to \mathcal{H}$ and a *-homomorphism $\pi : A \to B(\hat{\mathcal{H}})$ so that

$$\varphi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}}, \text{ for all } a \in A.$$

In other words, ϕ is a corner of a *-homomorphism

$$\pi(a) = \begin{pmatrix} \varphi(a) & * \\ * & * \end{pmatrix} \in B(\mathcal{H} \oplus (\mathcal{H}^{\perp}))$$

Remark

This generalizes the GNS construction for states.
There are many tweaks, special cases, and generalizations for this.

Big Facts

Theorem (Arveson's Extension Theorem)

Let A be a unital C^*-algebra with operator subsystem $\mathcal{S}$, and let $\mathcal{H}$ be any Hilbert space. Then any ccp map $\varphi : \mathcal{S} \to B(\mathcal{H})$ extends to a ccp map $\tilde{\varphi} : A \to B(\mathcal{H})$ so that $\tilde{\varphi}|_{\mathcal{S}} = \varphi$.

Remark

This generalizes the Hahn-Banach Theorem for linear functionals. There are many tweaks, special cases, and generalizations for this.

Definition

A C^*-algebra B is injective if for any C^*-algebra A with operator subsystem $\mathcal{S} \subseteq A$, any ccp map $\varphi : \mathcal{S} \to B$ extends to a ccp map $\tilde{\varphi} : A \to B$ so that $\tilde{\varphi}|_{\mathcal{S}} = \varphi$.

Example

- $B(\mathcal{H})$
- The hyperfinite II_1-factor $\mathcal{R} = \overline{\bigotimes_{n\in\mathbb{N}} M_2}^{\mathrm{WOT}}$

(Weak) Conditional Expectations

Definition (Tomiyama)

For C^*-algebras $A \subseteq B$, a conditional expectation is a ccp map $E : B \to A$ with $E|_A = \mathrm{id}_A$.

Proposition

For any discrete groups $\Lambda \leq \Gamma$, there is a canonical embedding $\mathrm{C}^*(\Lambda) \subseteq \mathrm{C}^*(\Gamma)$ and a conditional expectation $\mathrm{C}^*(\Gamma) \to \mathrm{C}^*(\Lambda)$.

Definition (Lance)

A weak conditional expectation is a ccp map $\varphi : B \to \pi_u(A)''$ such that $\varphi|_A = \pi_u$, or equivalently $\varphi : B \to A^{**}$ with $\varphi|_A = \iota_A$.

WEP/Relative Weak Injectivity

Definition

For C^*-algebras A and B with $A \subseteq B$, we say A embeds relatively weakly injectively (rwi) into B if there exsits a weak conditional expectation $B \to A^{**}$.

Definition (Lance)

We say a C^*-algebra A has the Weak Expectation Property (WEP) (or is weakly injective) if every embedding of A into another C^*-algebra is rwi.

Some quick reformulations

Proposition

Let A be a C^*-algebra. TFAE:

1. *A has the WEP.*[1]
2. *A embeds rwi into some injective C^*-algebra B.*
3. *A embeds rwi into $B(\mathcal{H}_u)$.*
4. *For every C^*-algebra B with $A \subseteq B$ and $*$-representation π of A there exists a ccp map $\varphi : B \to \pi(A)''$ such that $\varphi|_A = \pi$.*

[1] i.e. for any C^*-algebra $B \supseteq A$, $\exists$ ccp $\varphi : B \to \pi_u(A)''$, $\varphi|_A = \pi_u$.

Some first permanence properties

Proposition

The WEP is preserved a number of operations including

- *Tensoring with a nuclear* C^**-algebra [Lance]*
- *Direct products [Lance]*
- *Relatively weakly injective subalgebras [Lance]*
- *norm/weak closures of nested unions [Kirchberg]*

Group C^*-algebras?

We'll talk about $\mathrm{C}^*(\Gamma)$ later.

Recall:

A has we WEP iff for any C^*-algebra $B \supseteq A$ and any $*$-rep π of A there exists a ucp map $\rho : B \to \pi(A)''$ extending π.

Corollary

All traces on a unital C^*-algebra with WEP are amenable.

Corollary

A discrete group Γ is amenable[2] *iff $\mathrm{C}^*_\lambda(\Gamma)$ has WEP.*

[2]there is a state μ on $\ell^\infty\Lambda$ that is invariant by left translation: $\mu(g.f) = \mu(f),\ \forall\ g \in \Lambda, f \in \ell^\infty\Gamma$

Amenable Traces

Theorem (Connes, Kirchberg)

Let A be a unital C^*-algebra with a tracial state τ. TFAE:

1. There exits a net of ucp maps $\varphi_n : A \to M_{k_n}$ such that $\tau(a) = \lim_n \mathrm{tr}_{k_n} \circ \varphi_n(a)$ and for all $a, b \in A$
$$\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_{2,\mathrm{tr}_{k_n}} \to 0.$$
2. For any (some) embedding $A \subseteq B(\mathcal{H})$, there exists a ucp map $\rho : B(\mathcal{H}) \to \pi_\tau(A)''$ extending the GNS representation π_τ for τ.
3. For any (some) embedding $A \subseteq B(\mathcal{H})$ there exists a state $\varphi \in S(B(\mathcal{H}))$ such that $\varphi|_A = \tau$ and for every unitary $u \in \mathcal{U}(A)$ and $T \in B(\mathcal{H})$, we have
$$\varphi(uTu^*) = \varphi(T).$$

In any of these cases τ is called *amenable*.

QWEP

Definition

A C^*-algebra A is QWEP if it is the *quotient* of a WEP C^*-algebra.

Example

$$\mathcal{R}^\omega = \frac{\ell^\infty(\mathcal{R})}{\{(x_n) : \lim_{n\to\omega} \|x_n\|_{2,\tau} = 0\}}$$

for a non-principle ultrafilter ω on $\mathbb{N}$

I sold my soul to the devil, and the devil gave me an ultrafilter.

— Søren Eilers

Liftable Maps

Definition

Let A and B be C^*-algebras and $J \triangleleft B$ with $q : B \to B/J$. A ccp map $\varphi : A \to B/J$ is liftable if there exits a ccp map $\psi : A \to B$ with $\varphi = q \circ \psi$.

It is locally liftable if for any finite dimensional operator subsystem $\mathcal{S} \subseteq A$, $\varphi|_{\mathcal{S}}$ lifts to a ccp map $\psi : \mathcal{S} \to B$.

Remark

One can adjust a ccp map from a unital C^*-algebra to get a ucp map.

(L)LP

Definition

A unital C^*-algebra A has the (local) lifiting property (L)LP if any ucp map from A into a C^*-quotient (locally) lifts to a ucp map.

Remark

One can show that if all ucp maps lift to ucp maps, then all ccp maps lift to ccp maps.

$\mathrm{C}^*(\mathbb{F})$ has (L)LP

Theorem (Kirchberg)

For any free group $\mathbb{F}$, $\mathrm{C}^*(\mathbb{F})$ has the LLP. When $\mathbb{F}$ is countable, $\mathrm{C}^*(\mathbb{F})$ has the LP.

Lemma

Any unital *-homomorphism $\Phi : \mathrm{C}^*(\mathbb{F}_2) \to B/J$ is ucp liftable.

Proposition

For any $y \in B/J$ there exists $x \in B$ with $\|y\| = \|x\|$ and $q(x) = y$ where $q : B \to B/J$ is the quotient map.

Proposition (Halmos Dilation)

For a unital C^*-algebra A and $x \in A$ with $\|x\| \leq 1$,

$$\hat{x} = \begin{pmatrix} x & (1 - xx^*)^{1/2} \\ (1 - x^*x)^{1/2} & -x \end{pmatrix} \in \mathcal{U}(M_2(A)).$$

Remark

For a free group $\mathbb{F}$, any assignment of the generators of $\mathbb{F}$ to unitaries in a unital C^*-algebra A extends to *-homomorphism from $\mathrm{C}^*(\mathbb{F})$ onto the C^*-algebra generated by those unitaries.

Theorem (Kirchberg)

$\mathrm{C}^*(\mathbb{F}_2)$ *has the LP.*

Definition

The multiplier algebra $M(A)$ of a C^*-algebra A is the largest subalgebra of A^{**} containing A as an ideal: $A \triangleleft M(A) \subset A^{**}$.

Theorem (Kasparov's Stinespring Dilation Theorem)

Let A and D be unital C^*-algebras with A separable, and let $\phi : A \to D$ be a ucp map. Then there exists a unital *-homomophism $\Phi : A \to M(\mathcal{K} \otimes D)$ such that for each $a \in A$,

$$(e_{11} \otimes 1_B)\Phi(a)(e_{11} \otimes 1_B) = e_{11} \otimes \phi(a).$$

In particular, $\Phi(a)_{11} = \phi(a)$.

Theorem (Noncommutative Tietze Extension Theorem)

Any surjective *-homomorphism $\pi : C \twoheadrightarrow D$ between two C^*-algebras with σ-unital domain extends uniquely to a surjective *-homomorphism $\hat{\pi} : M(C) \to M(D)$.

Proposition

Let A be a unital C^*-algebra and $\mathbb{F}$ a free group with $I \lhd \mathrm{C}^*(\mathbb{F})$ such that $A \cong \mathrm{C}^*(\mathbb{F})/I$.
Then A has the LLP iff $\mathrm{id}_{\mathrm{C}^*(\mathbb{F})/I}$ locally lifts.
For A separable and $\mathbb{F}$ countable, A has the LP iff $\mathrm{id}_{\mathrm{C}^*(\mathbb{F})/I}$ lifts.

Permanence

Theorem

The LLP is preserved by a number of constructions including

- *Extensions of LLP by LLP [Kirchberg]*
- *Extensions of amenable groups by LLP groups [Fournier-Facio–Willett]*
- $\otimes$ *nuclear* C^**-algebras. (LP too) [Kirchberg]*
- *limits of nested unions of LLP (LP too with rwi inclusions) [Kirchberg]*
- *Full amalgamated free products over finite-dimensional* C^**-algebras (LP too) [Boca, Pisier, Ozawa, Shulman–Eilers]*

Example

$\mathrm{C}^*(\Gamma)$ has the LP where Γ is

- $\mathbb{Z}_m^{*k} = \overbrace{\mathbb{Z}_m * \ldots * \mathbb{Z}_m}^{k}$, for $k, m \geq 1$ [Boca]
- $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ [Kirchberg]
- Virtually free-by-cyclic [Fournier-Facio–Willett]
- Limit group [Fournier-Facio–Willett]
- Baumslag-Solitar group $BS(n, m)$ and the Baumslag-Gersten group [Fournier-Facio–Willett]
- Certain one-relator or right-angled Artin groups [Fournier-Facio–Willett]

Non-Examples for $\mathrm{C}^*(\Gamma)$

Theorem (Ozawa, Gromov)

There exists a group without LP.

Theorem (Thom)

Produced a group which is hyperlinear, property (T), and not RF.[3] *Hence no LLP.*

Theorem (Ioana–Spaas–Wiersma)

Let Γ be a discrete group.

- *If Γ contains $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ as a subgroup, (including $SL_n(\mathbb{Z}), n \geq 3$), then $\mathrm{C}^*(\Gamma)$ does not have the LLP.*
- *If Γ is countable with property (T) which is either not finitely presented or $H^2(\Gamma, \mathbb{Z}\Gamma) \neq 0$, then Γ does not have the LP.*

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[3] [Kirchberg] For Γ property (T), RF $\equiv \tau_\lambda \in T(\mathrm{C}^*(\Gamma))$ is amenable.

What about $\mathrm{C}^*_\lambda(\Gamma)$?

For Γ countable, discrete hyperlinear, Rădulescu's theorem says

$$\mathrm{C}^*_\lambda(\Gamma) \quad \subset \quad (L\Gamma, \tau_\lambda) \hookrightarrow \mathcal{R}^\omega$$

Theorem (Connes, Kirchberg)

Let A be a separable unital C^*-algebra with tracial state τ. TFAE:

1. *τ is amenable.*
2. *There exists a trace-preserving embedding $\pi_\tau(A)'' \subset \mathcal{R}^\omega$ such that the *-homomorphism $\pi_\tau : A \to \pi_\tau(A)'' \subseteq \mathcal{R}^\omega$ has a ucp lift $A \to \ell^\infty(\mathcal{R})$.*

Proposition (Kirchberg)

A ucp map $\varphi : A \to B/J$ from a separable C^*-algebra A with LLP into the quotient of a QWEP C^*-algebra B is ucp liftable.

Hard Questions

1. Does $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ have the (L)LP? (Ozawa)
2. Does LLP imply LP for separable C^*-algebras?
3. Are there any property (T) groups with the LLP?
4. For A separable, is LLP equivalent to $\mathrm{Ext}(A)$ being a group[4]?

[4] i.e., does every *-homomorphism $A \to B(\mathcal{H})/\mathcal{K}$ have a ccp lift

Epilogue: Stability

Classic: If two matrices almost commute, are they close to commuting operators?

Definition

For countable discrete Γ, an approximate morphism is a sequence of unital maps $\phi_n : \Gamma \to \mathcal{U}_{k_n}$ such that for all $g, h \in G$

$$\|\varphi_n(gh) - \varphi_n(g)\varphi_n(h)\|_\alpha \to 0$$

where $\|\cdot\|_\alpha$ is often the Hilbert-Schmidt $\|\cdot\|_{2,\mathrm{tr}_{k_n}}$ or operator norm.

Definition

Γ is Hilbert-Schmidt stable, resp. operator norm, stable if for any approximate morphism, there is a sequence of morphisms $\psi_n : \Gamma \to \mathcal{U}_{k_n}$ such that for all $g \in G$

$$\|\varphi_n(g) - \psi_n(g)\|_\alpha \to 0.$$

Approximate representations $\varphi_n \colon \Gamma \to \mathcal{U}(k_n)$	$\longleftrightarrow$	Unital $*$-homomorphisms $\Phi : C^*(\Gamma) \to \dfrac{\prod_n M_{k_n}}{\{\|x_n\|_\alpha \to 0 \text{ (along } \omega)\}}$

$\rightsquigarrow$ Stability says Φ lifts to a *-homomorphism $\Psi : \mathrm{C}^*(\Gamma) \to \prod_n M_{k_n}$.

Remark

There are many perturbations of stability including (very) flexibly stable where the LLP actually has something interesting to say.

$\mathrm{C}^*(\mathbb{F})$ has (L)LP

Theorem (Kirchberg)

For any free group $\mathbb{F}$, $\mathrm{C}^*(\mathbb{F})$ has the LLP. When $\mathbb{F}$ is countable, $\mathrm{C}^*(\mathbb{F})$ has the LP.

Proposition

For any discrete groups $\Lambda \leq \Gamma$, there is a canonical embedding $\mathrm{C}^*(\Lambda) \subseteq \mathrm{C}^*(\Gamma)$ and a conditional expectation $\mathrm{C}^*(\Gamma) \to \mathrm{C}^*(\Lambda)$. In particular this holds for $\mathbb{F}_2 \leq \mathbb{F}_n \leq \mathbb{F}_\infty \leq \mathbb{F}_2$.

Corollary

Let $\mathbb{F}$ be an arbitrary non-abelian free group, then for any finite dimensional subspace $D \subseteq \mathrm{C}^*(\mathbb{F})$, $\exists$ $\mathrm{C}^*(\mathbb{F}_\infty) \hookrightarrow \mathrm{C}^*(\mathbb{F})$ with

$$D \subseteq \mathrm{C}^*(\mathbb{F}_\infty) \subseteq \mathrm{C}^*(\mathbb{F})$$

and a conditional expectation $\mathrm{C}^*(\mathbb{F}) \to \mathrm{C}^*(\mathbb{F}_\infty)$.