Local Lifting and Weak Expectations

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Introduction

The following are (extended) notes from a YMC*A minicourse given in Oxford in August 2026 on Kirchberg's Local Lifting Property, Lance's Weak Expectation Property, and several of Kirchberg's seminal results tying the two together. Since his seminal work was published, several authors have expanded and simplified some of his arguments and results, notably Gilles Pisier ([43]), Narutaka Ozawa ([37]), and Nate Brown and Narutaka Ozawa([7]). We take advantage of this subsequent material for many of the arguments presented and will often refer the reader to these texts for proofs omitted here.

To keep me off tangents, I'll set a goal theorem.

Theorem (Kirchberg)**.** *The following are equivalent:*

1. *Every* II_1*-factor with separable predual embeds into some ultrapower* $\mathcal{R}^\omega$ *of the hyperfinite* II_1*-factor* $\mathcal{R}$*.*

2. *Every* C^**-algebra is QWEP (Kirchberg's QWEP Conjecture).*

3. $\mathrm{C}^*(\mathbb{F})$ *has WEP for any (every) non-abelian free group* $\mathbb{F}$*.*

4. $\mathrm{C}^*(\mathbb{F}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}) = \mathrm{C}^*(\mathbb{F}) \otimes_{\min} \mathrm{C}^*(\mathbb{F})$ *for any (every) non-abelian free group* $\mathbb{F}$*.*

5. $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ *is residually finite dimensional (RFD).*

6. *LLP* $\Rightarrow$ *WEP.*

Much of our work, however, will be to establish the following tensorial duality between the LLP and WEP:

Theorem (Kirchberg)**.** *Let* A *and* B *be* C^**-algebras. Then*

1. A *has the LLP* $\Leftrightarrow A \otimes_{\max} B(\ell^2) = A \otimes_{\min} B(\ell^2)$*,*

2. B *has the WEP* $\Leftrightarrow \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\max} B = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} B$*, and*

3. A *has the LLP and* B *has the WEP* $\Rightarrow A \otimes_{\max} B = A \otimes_{\min} B$*.*

References will be given to the authors of or inspirations for certain arguments as we go. We will often restrict arguments to a unital case and provide references (or wave our hands and say that it suffices to prove the claim for the unitization) for the non-unital case.

1 WEP (a.k.a. weak injectivity): A first look

Theorem 1.1 (Arveson). *Let A be a unital C^*-algebra operator subsystem (i.e., self-adjoint subspace) $\mathcal{S}$, and let $\mathcal{H}$ be any Hilbert space. Then any cp map $\varphi : \mathcal{S} \to B(\mathcal{H})$ extends to a cp map $\tilde{\varphi} : A \to B(\mathcal{H})$ so that $\tilde{\varphi}|_S = \varphi$.*

Remark 1.2. This also works with B and $\varphi/\ \tilde{\varphi}$ replaced with

- (non-unital) C^*-algebra and ccp maps
- operator subsystem and ucp maps
- operator subsystem and ccp map
- operator subspace (closed subspace) and cc maps
- operator subspace and cb maps

For more on this, see Section A.2 or more generally Appendix A.

What Arveson's theorem establishes is that $B(\mathcal{H})$ is injective– in several categories– but for the sake of streamlining, we'll say in the category of operator systems and ucp maps, where injectivity means exactly what the statement of the theorem says:

Definition 1.3. A C^*-algebra A is injective if for any C^*-algebra B with operator subsystem $\mathcal{S}$, any cp map $\psi : \mathcal{S} \to A$ extends to a cp map $\varphi : B \to A$ (meaning $\varphi|_{\mathcal{S}} = \psi$).[1]

Example 1.4. You know what else is injective? The hyperfinite II_1 factor

$$\mathcal{R} = \overline{\bigotimes_{\mathbb{N}} M_2}^{\mathrm{wot}}.$$

Injectivity is preserved under *conditional expectations*. Here we take the liberty (thanks to Tomiyama's theorem) to define a conditional expectation as follows:

Definition 1.5. For a C^*-algebra A and C^*-subalgebra B, a *conditional expectation* is a ccp map $E : A \to B$ with $E|_B = \mathrm{id}_B$.

Propostion 1.6. *If B is a C^*-subalgebra of an injective C^*-algebra A, then B is injective iff there exists a conditional expectation $A \to B$.*

Proof. If B is injective, then the identity on B extends to a conditional expectation. On the other hand, if there is such a conditional expectation, then the proof is following diagram.

$$\begin{tikzcd} C \arrow[r, "\text{Arveson's}"] & A \arrow[ddr, "ccp"] & \\ \cup & \cup & \\ D \arrow[r, "ucp"] & B \arrow[r, "\mathrm{id}"] & B \end{tikzcd}$$

□

[1] We will usually use this to argue that ccp/ ucp maps always extend to ccp/ucp maps. Given that we will be working with unital C^*-algebras, this special case follows from the fact that $\|\varphi\|_{cb} = \|\varphi(1)\|$ for cp maps.

Definition 1.7. For a C*-algebra A and C*-subalgebra B, a **weak** *conditional expectation* is an ccp map $A \to \overline{\pi_u(A)}^{\text{wot}}$, which when restricted with B agrees with the universal representation $\pi_u : B \to B(\mathcal{H}_u)$.

Remark 1.8. Since we are sticking to the unital setting, we'll also want our subalgebras and embeddings to be unital with the same unit ($1_A = 1_B$).

Remark 1.9. This is often rephrased using the identification $\overline{\pi_u(A)}^{\text{wot}} = \pi_u(B)'' \cong B^{**}$.

Definition 1.10. Let A be a C*-algebra. If A embeds into a C*-algebra B such that there is a weak conditional expectation $B \to A^{**}$, then we say A embeds relatively weakly injectively (rwi) into B (or A is weakly cp complemented in B).

Example 1.11. $A \subseteq A^{**}$ is relatively weakly injective with weak conditional expectation $\mathrm{id}_{A^{**}}$.

Propostion 1.12. *If $A \subseteq B$ is a hereditary subalgebra (e.g. a two-sided closed ideal), then $A \subseteq B$ relatively weakly injectively.*

Proof. As a hereditary[2] subalgebra A has an approximate identity $\{e_i\}_{i \in I}$. Define maps $\varphi_i : B \to A$ by $\varphi_i(b) = e_i b e_i$. Extending the co-domain gives us a family of uniformly bounded linear maps into A^{**}, which has a weak* cluster point $\Phi : B \to A^{**}$ (see e.g. [7, Theorem 1.3.7]). Since the φ_i converge pointwise in norm to id_A, Φ is a weak conditional expectation. □

Definition 1.13 (Lance)**.** If every embedding of A into any other C*-algebra is weakly cp complemented, then A is said to have the *Weak Expectation Property* (WEP) (or A is *weakly injective.*

Example 1.14. Every nuclear[3] C*-algebra has the WEP.[4]

Example 1.15 ([43])**.** Moreover, a von Neuamnn algebra M has the WEP iff it is injective (iff it is hyperfinite [10]).

Proof. Injectivity is generally stronger than the WEP. To see why $M \subseteq B(\mathcal{H})$ having the WEP is sufficient for injectivity, assume $B \subseteq A$ are C*-algebras $\phi : B \to M$ is ucp. Then, by the universality of M^{**}, we have the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\text{Arveson's}"] \arrow[d, phantom, "\cup"] & B(\mathcal{H}) \arrow[d, phantom, "\cup"] \arrow[dr, "WEP"] & & \\
B \arrow[r, "\phi"] & M \arrow[r, hookrightarrow] & M^{**} \arrow[r, twoheadrightarrow] & M'' = M
\end{tikzcd}$$

Composing these maps, we get a conditional expectation $B(\mathcal{H}) \to M$. □

[2] For any $0 \leq a \in A$, if there exists $0 \leq b \in B$ with $b \leq a$, then $b \in A$.

[3] A C*-algebra A is nuclear if there exist ccp maps $A \xrightarrow{\psi_n} M_{k_n} \xrightarrow{\varphi_n} A$ such that $\|\varphi_n(\psi_n(a)) - a\| \to 0$ for all $a \in A$. By a well-known result of Kirchberg and Choi–Effros, this is equivalent to the tensor product of A with any other C*-algebra having a unique C*-norm.

[4] One can prove this now using the CPAP characterization of nuclearity along with Arveson's extension theorem and point-ultraweak limits of nets. The proof will get much easier after we see Lance's characterization of WEP.

Sometimes it is useful to consider double commutants instead of the double dual:

Propostion 1.16. *Let A be a* C**-algebra. TFAE:*

1. *A has the WEP.*
2. *A embeds relatively weakly injectively into some injective* C**-algebra B.*
3. *There exists a ccp map $\varphi : B(\mathcal{H}_u) \to \pi_u(A)''$ such that $\varphi|_{\pi_u(A)} = id_{\pi_u(A)}$.*
4. *For every* C**-algebra B such that A embeds into B and any $*$-homomorphism $\pi : A \to B(\mathcal{H})$ there exists a ccp map $\rho : B \to \pi(A)''$ such that $\rho(a) = \pi(a)$ for all $a \in A$.*

Remark 1.17. This is an adaptation of Prop 3.6.6 in [7], which also shows that (2) $\Longleftrightarrow$ (3) holds for any fixed C*-inclusion $A \subseteq B$.

Proof. For (1) $\Rightarrow$ (4) we use the universal property of $A^{**} \simeq \pi_u(A)''$: Suppose $A \subseteq B$ with weak conditional expectation $B \to A^{**}$. Composing this with the normal extension $\tilde{\pi} : A^{**} \to \pi(A)''$ (guaranteed by universality of A^{**}) of $\pi : A \to B(\mathcal{H})$ yields the desired map ρ. Clearly (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2) by Arveson's Theorem. To see (2) $\Rightarrow$ (1), we use the following diagram

$$\begin{tikzcd}
& B \arrow[dr, "\text{r.w.i.}"] & \\
C \arrow[ur, "\text{injectivity}"] \arrow[r, phantom, "\supseteq"] & A \arrow[u, hook] \arrow[r, hook] & A^{**}
\end{tikzcd}$$

□

So what are some examples beyond injective and nuclear C*-algebras?
Here are some permanence properties of WEP:

Propostion 1.18. *The WEP is preserved under the following:*

- *Tensoring with a nuclear* C**-algebra (Lance)*[5].
- *Direct products (Lance).*
- *Amenable crossed products (Buss–Echterhoff–Willett)* [6]

Definition 1.19 (Kirchberg)**.** A C*-algebra is *QWEP* if it is the quotient of a WEP C*-algebra.

Here's the only example you really need to know.

[5] Kirchberg says it explicitly, but Lance did the footwork

[6] This was first proved for a stronger notion of amenable actions (the one found in [7]) by Bhattacharya–Farenick. The more widely accepted definition of amenable actions (and its equivalent formulations) can be found here [39].

Example 1.20. Since $\mathcal{R}$ is injective, $\ell^\infty(\mathcal{R})$ is WEP. Hence for any (non-principal) ultrafilter ω on $\mathbb{N}$, the ultrapower of the hyperfinite II_1-factor

$$\mathcal{R}^\omega := \frac{\ell^\infty(\mathcal{R})}{\{(a_n)_n \mid \lim_{n\to\omega} a_n = 0\}}$$

is QWEP.

One of the ingredients to Connes' remarkable classification result [10] was the notion of amenable traces. There are several equivalent formulations; we provide three here. This is due to Connes in the unique trace case and Kirchberg in general. For the proof, which is rather technical and delicate, see [7, Chapter 6.2].

Theorem 1.21 (Connes, Kirchberg). *Let A be a unital C^*-algebra with a tracial*[7] *state τ. Then the following are equivalent:*

1. *For any (some) embedding $A \subseteq B(\mathcal{H})$ there exists a state $\varphi \in S(B(\mathcal{H}))$ such that $\varphi|_A = \tau$ and for every unitary $u \in \mathcal{U}(A)$ and $T \in B(\mathcal{H})$, we have*
$$\varphi(uTu^*) = \varphi(T).$$
2. *There exits a net of ucp maps $\varphi_n : A \to M_{k_n}$ such that $\tau(a) = \lim_n tr_{k_n} \circ \varphi_n(a)$ and $\|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_{2,tr_{k_n}} \to 0$ for all $a, b \in A$.*[8]
3. *For any (some) faithful representation $A \to B(\mathcal{H})$, there exists a ucp map $\Phi : B(\mathcal{H}) \to \pi_\tau(A)''$ such that $\Phi(a) = \pi_\tau(a)$ for all $a \in A$.*
4. *(For A separable) There exists a trace-preserving embedding $\pi_\tau(A)'' \subset \mathcal{R}^\omega$ such that the *-homomorphism $\pi_\tau : A \to \pi_\tau(A)'' \subseteq \mathcal{R}^\omega$ has a ucp lift $A \to \ell^\infty(\mathcal{R})$.*

Corollary 1.22. *If a unital C^*-algebra A has WEP, then any tracial state on A is amenable.*

In particular, if the reduced C^*-algebra, $\mathrm{C}^*_\lambda(\Gamma)$, of a discrete group Γ has WEP, then Γ is amenable.

The first follows directly from the previous characterization of WEP, but for the sake of lecture prep, I'll write an argument skipping that.

Proof. Let τ be a tracial state on $A \subseteq B(\mathcal{H})$. Then by WEP, there exists a ucp map $\varphi : B(\mathcal{H}) \to A^{**}$ extending the embedding $A \subseteq B(\mathcal{H})$. By universality, π_τ has a normal extension to $\pi_\tau'' : A^{**} \to \pi_\tau(A)''$. Let $\Phi = \pi_\tau'' \circ \varphi$.

The second follows from the fact that a discrete group Γ is amenable if[9] it has an amenable trace. Indeed, given $\mathrm{C}^*_\lambda(\Gamma) \subset B(\ell^2\Gamma))$, the state $\varphi \in S(B(\ell^\infty))$ given by (1) restricts to an invariant mean on $\ell^\infty\Gamma$.[10] □

[7] Tracial means $\tau(ab) = \tau(ba)$ for all $a, b \in A$.

[8] Here tr_{k_n} is the normalized trace on M_{n_k}, and $\|x\|_{2,\mathrm{tr}_{k_n}} = \mathrm{tr}_{k_n}(x^*x)^{1/2}$.

[9] In fact, this is an iff. See [7, Proposition 6.3.3] for a full proof.

[10] An invariant mean is a state μ on $\ell^\infty\Gamma$ which is invariant under left translation: for $g \in \Gamma$ and $f \in \ell^\infty$, $\mu(g.f) = \mu(f)$. Note that in $B(\ell^2\Gamma)$, left translation is spatially implemented, i.e., $g.f = \lambda_g f \lambda_g^*$.

2 The LLP: A first look

Definition 2.1. Let A and B be unital C*-algebras and J a closed two-sided ideal in B with quotient map $\pi : B \to B/J$. A ucp map $\phi : A \to B/J$ is **liftable** if there is a ucp map $\psi : A \to B$ such that $\pi \circ \psi = \phi$. We say $\phi : A \to B/J$ is **locally liftable** if for any finite dimensional operator system $S \subseteq A$ there is a ucp map $\psi : S \to B$ such that $\pi \circ \psi = \phi|_S$.

Definition 2.2 (LLP)**.** A unital C* algebra A has the **(local) lifting property** (L)LP if any ucp map from A into a quotient C*-algebra is (locally) liftable. (A non-unital C*-algebra has (L)LP iff its unitization does.)

Remark 2.3. Technically, "liftable" is defined for ccp maps and, when the algebras in question are unital, ucp maps are required to lift to ccp. However, a ccp lifts of a ucp map can be adjusted to a ucp lift via the standard trick of adding $\rho(\cdot)(1 - \varphi(1))$ for some state ρ which is 1 on $\varphi(1)$, and by a similar trick and the fact (due to Arveson) that the set of liftable ccp maps from a separable operator system into a C*-quotient is point-norm closed, asking that every ccp map lifts to a ccp map can be reduced to asking the same for ucp maps ([7, 13.1.2]), and . So we only consider ucp maps here.

Example 2.4. The Choi-Effros lifting theorem tells us that all nuclear C*-algebras have the LLP (LP when they are separable).

Remark 2.5. Without separability assumptions, LLP might be the best one can hope for. Consider the nuclear C*-algebra $c_0(\mathbb{R})$ (with $\mathbb{R}$ considered as a discrete set). This embeds into $\ell^\infty(\mathbb{N})/c_0(\mathbb{N})$ because the latter has uncountably many orthogonal projections (coming from indicator functions χ_A for subsets of $\mathbb{N}$ with finite intersection) to which we can map the indicator functions $\delta_t \in c_0(\mathbb{R})$. If this embedding lifted to a ccp map, we would in particular have a bounded linear embedding $c_0(\mathbb{R}) \hookrightarrow \ell^\infty(\mathbb{N})$. Since $\ell^\infty(\mathbb{N})$ has a separable pre-dual, it's points can be separated by countably many linear functionals, which will also hold for any subspace, i.e., $c_0(\mathbb{R})$ is separated by countably many linear functionals. This is a contradiction (since any $\varphi \in c_0(\mathbb{R})^* \cong \ell^1(\mathbb{R})$ has countable support).

One can think of the LP property as "ucp projectivity" (much in the same way as "injectivity" for C*-algebras is actually ucp injectivity). In that light, the (separable case of) the following theorem, should be thought of as Kirchberg's projective analogue to Arveson's Hahn-Banach Extension Theorem.

Theorem 2.6. *[30] For any free group $\mathbb{F}$, $\mathrm{C}^*(\mathbb{F})$ has the LLP. If $\mathbb{F}$ is countably generated, then $\mathrm{C}^*(\mathbb{F})$ has the LP.*

The proof follows the sketch outlined in [7, 13.1.3]. Before we get into it, let's just get one thing straight– as far as the theory at hand is concerned, different free groups will be interchangeable. Hence why you sometimes see all the statements in terms of $\mathbb{F}_\infty$ or $\mathbb{F}_2$. This is due to the following lemma/ corollary. (How this gets used in each instance is a good exercise which will soon feel repetitive.)

Lemma 2.7. *[43, Prop 8.8] For groups $\Lambda \subseteq \Gamma$, there exists a canonical embedding* $\mathrm{C}^*(\Lambda) \hookrightarrow$

$\mathrm{C}^*(\Gamma)$ *and a conditional expectation from* $\mathrm{C}^*(\Gamma)$ *onto the image of* $\mathrm{C}^*(\Lambda)$ *under this embedding.*[11]

Note that $\mathbb{F}_2$ contains a copy of $\mathbb{F}_\infty$ as a subgroup (in fact, its commutator subgroup), so we've got embeddings and conditional expectations among all the separable $\mathrm{C}^*(\mathbb{F})$. For the nonseparable case, let's go ahead and flesh out the relevant corollary (which was given in [43] but not explicitly as a corollary).

Corollary 2.8. *Let* $\mathbb{F}$ *be an arbitrary free group, then for any finite dimensional subspace* $D \subseteq \mathrm{C}^*(\mathbb{F})$*, there is a* C^**-subalgebra* C *of* $\mathrm{C}^*(\mathbb{F})$ *containing* D*, which is* **-isomorphic to* $\mathrm{C}^*(\mathbb{F}_\infty)$*, and a conditional expectation* $\mathrm{C}^*(\mathbb{F}) \to C$*.*

Proof Sketch. Any element in $\mathrm{C}^*(\mathbb{F})$ can be expressed with countably many elements $\{g_i\}_{i=1}^\infty$ in $\mathbb{F}$. Then we find a copy of $\mathbb{F}_\infty$ in $\mathbb{F}$ containing $\{g_i\}_{i=1}^\infty$. This will induce an embedding $\mathrm{C}^*(\mathbb{F}_\infty) \hookrightarrow \mathrm{C}^*(\mathbb{F})$, and the rest follows from Lemma 2.7. □

Lemma 2.9. *If* $\mathrm{C}^*(\mathbb{F}_2)$ *has the LP, then* $\mathrm{C}^*(\mathbb{F}_n)$ *has the LP for all countable free groups, and* $\mathrm{C}^*(\mathbb{F})$ *has the LLP for all free groups* $\mathbb{F}$*.*

Proof. Assume $\mathrm{C}^*(\mathbb{F}_2)$ has the LP. Fix B unital with $J \triangleleft B$ and $q : B \to B/J$.

We prove the claim for separable free groups. Since $\mathbb{F}_n \le \mathbb{F}_\infty \le \mathbb{F}_2$, the claim for the others will follow suit. Write $E : \mathrm{C}^*(\mathbb{F}_2) \to \mathrm{C}^*(\mathbb{F}_\infty)$ for the conditional expectation guaranteed by Lemma 2.7.

Let $\phi : \mathrm{C}^*(\mathbb{F}_\infty) \to B/J$ be ucp. Then $\phi \circ E : \mathrm{C}^*(\mathbb{F}_2) \to B/J$ lifts to a ucp map $\widetilde{\phi \circ E} : \mathrm{C}^*(\mathbb{F}_2) \to B$. The restriction of this map to $\mathrm{C}^*(\mathbb{F}_\infty) \subseteq \mathrm{C}^*(\mathbb{F}_2)$ is the desired lift of ϕ.

Now, let $\mathbb{F}$ be an arbitrary free group and $\varphi : \mathrm{C}^*(\mathbb{F}) \to B/J$. For any finite-dimensional subspace $D \subseteq \mathrm{C}^*(\mathbb{F})$, we find a copy of $\mathrm{C}^*(\mathbb{F}_\infty)$ in $\mathrm{C}^*(\mathbb{F})$ so that $D \subseteq \mathrm{C}^*(\mathbb{F}_\infty) \subseteq \mathrm{C}^*(\mathbb{F})$. Then $\varphi|_{\mathrm{C}^*(\mathbb{F}_\infty)}$ is liftable by our previous argument, and hence so is its restriction to the smaller space D. □

Lemma 2.10. *Let* $\mathbb{F}_n$ *be a free group on* n *generators for* $n = 2, 3, ..., \infty$*. Any* **-homomorphism from* $\mathrm{C}^*(\mathbb{F}_n)$ *into a* C^**-quotient* B/J *is liftable to a ucp map* $\mathrm{C}^*(\mathbb{F}_n) \to B$*.*

Proof. We do the proof for $n = 2$. The others are the same. Let B be a unital C^*-algebra with closed two-sided ideal J and quotient map $q : B \to B/J$, and let $\pi : \mathrm{C}^*(\mathbb{F}_2) \to B/J$ a *-homomorphism. Write U_1, U_2 for the canonical generators of $\mathrm{C}^*(\mathbb{F}_2)$. Then $\pi(U_1), \pi(U_2)$ are contractions in B/J and so they lift to contractions $x_1, x_2 \in B$ (i.e., $\|x_i\| \le 1$ and $q(x_i) = \varphi(U_i)$). Then the Halmos dilations

$$\hat{x}_i = \begin{pmatrix} x_1 & (1 - x_i x_i^*)^{1/2} \\ (1 - x_i^* x_i)^{1/2} & -x_i^* \end{pmatrix} \in M_2(B)$$

[11] Proof idea: This essentially comes from the induced representations $\mathrm{Ind}_\Lambda^\Gamma(\pi)$ on Γ coming from representations π of Λ. When restricted to G, the conditional expectation is just the indicator function on Λ.

are unitaries. By the universal property of $\mathrm{C}^*(\mathbb{F}_2)$, the assignment $U_i \mapsto \hat{x}_i$ induces a *-homomorphism $\mathrm{C}^*(\mathbb{F}_2) \to \mathrm{C}^*(\hat{x}_1, \hat{x}_2)$. Composing this *-homomorphism with the compression of $M_2(B)$ onto its 1-1 corner (a ucp map), given a lift of φ.[12] $\square$

Proof of Theorem 2.6. We start with the case that $\mathrm{C}^*(\mathbb{F})$ is separable, where we want to show the LP. Let B be a unital C^*-algebra and $J \lhd B$, and let $\phi : \mathrm{C}^*(\mathbb{F}) \to B/J$ be ucp. Since $\mathrm{C}^*(\mathbb{F})$ is separable, Kasparov's Stinespring Dilation Theorem[13] then yields a *-homomorphism $\Phi : \mathrm{C}^*(\mathbb{F}) \to M(\mathcal{K} \otimes B/J)$ so that such that $(\Phi(a))_{11} = \phi(a)$ for all $a \in \mathrm{C}^*(\mathbb{F})$, where we have identified B with $e_{11} \otimes B \subset \mathcal{K} \otimes B \subset M(\mathcal{K} \otimes B)$ and likewise B/J with $e_{11} \otimes B/J \subset M(\mathcal{K} \otimes B/J)$. Since B is unital, $\mathcal{K} \otimes B/J$ is σ-unital, and thus we can apply the noncommutative Tietze Extension Theorem[14] to $\mathrm{id}_{\mathcal{K}} \otimes q : \mathcal{K} \otimes B \to \mathcal{K} \otimes B/J$ to get a surjection $M(\mathcal{K} \otimes B) \to M(\mathcal{K} \otimes B/J)$, which we also call $\hat{\pi}$. Note that for each $x \in M(\mathcal{K} \otimes B)$, we have $(\hat{\pi}(x))_{11} = \pi(x_{11})$. For those keeping score, this gives us the following commutative diagram

$$\begin{tikzcd}
 & M(\mathcal{K} \otimes B) \arrow[d, "\hat{\pi}"] \arrow[dr] & \\
 & M(\mathcal{K} \otimes B/J) \arrow[dr] & B \arrow[d, "\pi"] \\
\mathrm{C}^*(\mathbb{F}) \arrow[ur, "\Phi"] \arrow[rr, "\phi"] & & B/J.
\end{tikzcd}$$

Now by Lemma 2.10, we can lift Φ giving us the following commutative diagram and rounding off the proof.

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[12] This is something to check, but it will fall out when the $(1 - x_i x_i^*)^{1/2}$ and $(1 - x_i^* x_i)^{1/2}$ terms in words go through q.

[13]

Theorem 2.11 (Kasparov's Stinespring Dilation Theorem)**.** *Let A and B be unital* C^**-algebras with A separable, and let $\phi : A \to B$ be a ucp map. Then there exists a unital *-homomophism $\Phi : A \to M(\mathcal{K} \otimes B)$ such that for each $a \in A$,*

$$(e_{11} \otimes 1_B)\Phi(a)(e_{11} \otimes 1_B) = e_{11} \otimes \phi(a).$$

In particular, $\Phi(a)_{11} = \phi(a)$.

The above is not *precisely* what Kasparov showed in [27, Theorem 3]; however it is a corollary to the proof. For a proof, see [13].

[14]

Theorem 2.12 (Noncommutative Tietze Extension Theorem)**.** *Let A and B be* C^**-algebras and $\pi : B \to A$ a surjective *-homomorphism. Then π extends uniquely to a *-homomorphism $\hat{\pi} : M(B) \to M(A)$ where $\hat{\pi}(x) \cdot \pi(b) = \pi(x \cdot b)$ for all $b \in B$. Moreover, if B is σ-unital, then π is surjective.*

This was first proved for separable C^*-algebras in [1, Theorem 4.2]. For the σ-unital case, the reader is referred to [33, Proposition 6.8] or [34, Theorem 9.2.1], both of which follow the argument for separable C^*-algebras in [41, Proposition 3.12.10].

$$\begin{tikzcd}
 & M(\mathcal{K}\otimes B) \arrow[d, "\hat{\pi}"] \arrow[dr] & \\
 & M(\mathcal{K}\otimes B/J) \arrow[dr] & B \arrow[d, "\pi"] \\
\mathrm{C}^*(\mathbb{F}) \arrow[uur, bend left, "\theta"] \arrow[ur, "\Phi"] \arrow[rr, "\phi"] & & B/J,
\end{tikzcd}$$

For an uncountable free group $\mathbb{F}$, we show the LLP. But since this allows us to restrict to finite-dimensional subsystems, by Corollary 2.8, this is covered by the LP for $\mathrm{C}^*(\mathbb{F}_\infty)$. $\square$

Propostion 2.13. *Let A be a C^*-algebra and $\mathbb{F}$ a free group such that A can be identified with a quotient $\mathrm{C}^*(\mathbb{F})/J$ of $\mathrm{C}^*(\mathbb{F})$. Then A has the LLP iff the identity on $\mathrm{C}^*(\mathbb{F})/J$ is locally liftable. If A is separable, then $\mathbb{F}$ can be chosen to be separable, and A has the LP iff the identity on $\mathrm{C}^*(\mathbb{F})/J$ is liftable.*

Proof. For simplicity, identify $A = \mathrm{C}^*(\mathbb{F})/J$, and let $\pi : \mathrm{C}^*(\mathbb{F}) \to \mathrm{C}^*(\mathbb{F})/J$ be the quotient map. Let $E \subseteq A$ be a finite dimensional operator system and $\rho : E \to \mathrm{C}^*(\mathbb{F})$ the lift of $\mathrm{id}_A|_E$ guaranteed by assumption. Let B a C*-algebra with closed two-sided ideal I, and $\varphi : A \to B/I$ a ucp map. Then, $\varphi \circ \pi : \mathrm{C}^*(\mathbb{F}) \to B/I$ is a ucp map and $\rho(E) \subseteq \mathrm{C}^*(\mathbb{F})$ is a finite dimensional operator system. Then, since $\mathrm{C}^*(\mathbb{F})$ has the LLP, there is a lift $\widetilde{(\varphi \circ \pi)}|_{\rho(E)}$ of $(\varphi \circ \pi)|_{\rho(E)}$ to B.

$$\begin{tikzcd}
 & & & B \arrow[dd, two heads] \\
\rho(E) \arrow[urrr, "\widetilde{(\varphi\circ\pi)}|_{\rho(E)}"] & \subseteq & \mathrm{C}^*(\mathbb{F}) \arrow[d, two heads, "\pi"] \arrow[dr, "\varphi\circ\pi"] & \\
E \arrow[u, "\rho"] & \subseteq & A \arrow[r, "\varphi"] & B/I
\end{tikzcd}$$

Having seen the local version, the separable case should be easy to sort. $\square$

The upshot of this corollary is that, to show that any ucp map lifts, we need only show that one ucp map (in fact *-homomorphism) lifts. With this, we can give a class of examples.

Remark 2.14. There is often interest in when one can go from local to global lifts. In fact, it is open whether the LLP implies the LP for separable C*-algebras.[15]

However we do have the following due to Kirchberg (see also [37, Corollary 3.12]).

Propostion 2.15. *[29] A ucp map $\varphi : A \to B/J$ from a separable C^*-algebra A with the LLP into the quotient of a QWEP C^*-algebra B is ucp liftable.*

[15] This would have been a consequence of CEP if true. By the end of this lecture series, try to see if you can figure out why.

I would be remiss to not mention the converse, due to Ozawa, which I and others have used.

Propostion 2.16. *[36] A separable* C**-algebra A has the LLP iff any ucp map from A into the Calkin algebra is ucp liftable.*

2.1 Permanence and Examples

Many of these properties and further permanence properties are laid out nicely in the recent article by Fournier-Facio and Willett [19].

Exercise 2.17. If B is a C*-algebra with the (L)LP, and $A \subset B$ is a C*-subalgebra with a conditional expectation $B \to A$, then A has the (L)LP. In particular, if Γ and Λ are discrete groups with $\Lambda \leq \Gamma$, if C$^*(\Gamma)$ has the (L)LP, then so does C$^*(\Lambda)$.

Remark 2.18. This holds in more generality: for weak conditional expectations ([29, Corollary 2.6(v)]). We'll come back to this at Proposition 5.9.

Propostion 2.19. *[29, Corollary 2.6].*

- *The LLP is preserved by taking extensions.*
- *The (L)LP is preserved by tensoring with a nuclear* C**-algebra.*
- *The LLP is preserved under taking limits of nested unions of* C**-algebras. The same holds for LP with relative weakly injective inclusions.*

Theorem 2.20. *[8, Theorem 1.16] If a discrete group Γ acts amenably on a* C**-algebra A, then A has the LLP if and only if $A \rtimes_{\max} \Gamma$ is.*

Theorem 2.21 (Pisier, Boca, Ozawa, Shulman–Eilers)**.** *The (L)LP is closed under taking amalgamated free products over finite-dimensional* C**-algebras.*

Boca proved the LP is closed under free products [6], Pisier proved it for the LLP [43], which Ozawa generalized to the LLP for amalgamated products [37]. We will see the main ingredients of Pisier's proof later.

Example 2.22. The following C*-algebras have the LP

- C$^*(\mathbb{Z}_m^{*k})$, $k, m \geq 1$
- C$^*(\mathrm{SL}_2(\mathbb{Z})) \cong$ C$^*(\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6)$.[16]

By establishing that LLP is preserved by certain central/amenable group extensions, Fournier-Facio and Willett were also able to show the LP holds for the following classes of finitely generated groups [19].

- Fundamental groups of connected dim ≤ 3 Manifolds

[16] This was already well known by Kirchberg's work – likely using the fact that $\mathbb{F}_2 \leq \mathrm{SL}_2(\mathbb{Z})$ is a finite index subgroup.

- Virtually free-by-cyclic
- Certain one-relator groups
- Limit groups
- Certain right-angled Artin groups
- Baumslag-Solitar groups $BS(n, m)$ and the Baumslag-Gersten group.

2.2 Examples and Non-examples

Outside the nuclear setting and easy corollaries to the above, examples and non-examples of the LLP are hard to come by, and there is significant interest in when this property holds/ fails – particularly for *full* group C*-algebras.

Remark 2.23. What about reduced group C*-algebras $\mathrm{C}^*_\lambda(\Gamma)$? Do any of the non-nuclear ones have the (L)LP?

Not if they are hyperlinear (embed into the unitary group $\mathcal{U}(\mathcal{R}^\omega)$ of an ultrapower of the hyperfinite II$_1$-factor). Indeed, by Rădulescu's theorem, hyperlinearity of Γ is equivalent to having a trace-preserving embedding of the group von Neumann algebra $L\Gamma \hookrightarrow \mathcal{R}^\omega$, which then restricts to a trace-preserving *-homomorphism from the reduced C*-algebra $\phi : \mathrm{C}^*_\lambda(\Gamma) \to \mathcal{R}^\omega$.

Now if $\mathrm{C}^*_\lambda(\Gamma)$ had the (L)LP, then ϕ would lift[17] to a ucp map $\tilde{\phi} : \mathrm{C}^*_\lambda(\Gamma) \to \ell^\infty(\mathcal{R})$. Then, by definition the canonical trace τ_λ on $\mathrm{C}^*_\lambda(\Gamma)$ would be amenable (Theorem 1.21). But this is equivalent to Γ being amenable (see [7, Proposition 6.3.3]). Hence, if τ_λ were amenable, then $\mathrm{C}^*_\lambda(\Gamma)$ would have the (L)LP. So for hyperlinear groups the (L)LP of $\mathrm{C}^*_\lambda(\Gamma)$ is equivalent to the amenability of Γ.

In [38], Ozawa proved the existence of groups whose full C*-algebra does not have the LP, but no concrete examples failing the LLP were known until Andreas Thom's 2008 example ([49]) of hyperlinear proptery (T) groups that are not residually finite. For the sake of exposition, here's why these lack the LLP.

Example 2.24 (Thom)**.** Let Γ be a discrete hyperlinear group, i.e., $\mathrm{C}^*_\lambda(\Gamma)$ has a trace-preserving embedding into $\mathcal{R}^\omega$. Then pre-composing with the canonical surjection $\mathrm{C}^*(\Gamma) \to \mathrm{C}^*_\lambda(\Gamma)$, this gives a *-homomorphism $\mathrm{C}^*(\Gamma) \to \mathcal{R}^\omega$ that preserves the trace τ_λ on $\mathrm{C}^*(\Gamma)$ coming from the left-regular representation (i.e., the trace on $\mathrm{C}^*_\lambda(\Gamma)$).

On the other hand, as Kirchberg showed in [31], when a group Γ has property (T), then residual finiteness (RF) is the same as Kirchberg's factorization property (F) (the trace τ_λ on $\mathrm{C}^*(\Gamma)$ coming from the left-regular representation is amenable (see [7, Theorem 6.4.3]).

In [49], Thom produced a group which is hyperlinear, property (T), and not RF. Hence for this Γ, the τ_λ-preserving *-homomorphism $\mathrm{C}^*(\Gamma) \to \mathcal{R}^\omega$ from hyperlinearity cannot lift.

There's been a buzz around the (L)LP recently, particularly in how it relates to group stability. For example, groups whose full C*-algebras have the LLP and are RFD are are

[17] If $\mathrm{C}^*_\lambda(\Gamma)$ has only the LLP, then this uses Proposition 2.15.

very flexibly stable[18] with respect to the operator norm [19], resp. the Hilbert-Schmidt norm [14]. See [19, Section 7] and [14] for a further discussion on LLP's relation to various stability properties.

Here are some recent highlights.

Example 2.25 (Ioana–Spaas–Wiersma)**.** [23] Let Γ be a discrete group.

1. If Γ is countable with property (T) and $H^2(\Gamma, \mathbb{Z}\Gamma) \neq 0$ (nonvanishing second cohomology group), then Γ does not have the LP.

2. If Γ is a non-finitely presented countable property (T) group, then Γ does not have the LP.

3. If Γ contains $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ as a subgroup, (including $\mathrm{SL}_n(\mathbb{Z}), n \geq 3$), then $\mathrm{C}^*(\Gamma)$ does not have the LLP.[19]

The first examples stem from the more general [23, Theorem F] for the first, [23, Corollary D] for the second, and [23, Theorem A] for the third.

2.3 Open Questions

These are some of the most commonly asked questions around the LLP. For some more low-hanging or stability-flavored fruit, check out [19].

1. Does $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ have the (L)LP? (Ozawa)

2. Are there any (relative) property (T) groups with the LLP?

3. Does LLP imply LP for separable C*-algebras?[20]

4. For A separable, is LLP equivalent to $\mathrm{Ext}(A)$ being a group?[21]

[18] This notion of very flexible X stability (where X is Hilbert-Schmidt or operator norm) goes back to Becker and Lubotzky in [2] and is fleshed out nicely here [14]. Essentially it means any approximate representation is approximately close to a corner of a representation.

An approximate representation is sequence of unital maps $\{\Gamma \to \mathcal{U}_{k_n}\}_n$ that are pointwise approximately multiplicative (where "approximately" is interpreted in the operator or Hilbert-Schmidt norm). Note that this translates to a *-homomorphism from $\prod_\omega M_{k_n}$, resp. $\mathrm{C}^*(\Gamma) \prod^\omega M_{k_n}$ for some ultrafilter ω (where $^\omega$ indicates the HS norm ultraproduct and $_\omega$ the operator norm ultraproduct.

A group is is HS, resp. operator norm, stable if for any approximate representation, there's a sequence of honest representations $\{\Gamma \to \mathcal{U}_{k_n}\}_n$ that get pointwise arbitrarily close to our approximate representations in HS, resp. operator, norm. Or more cleanly, every *-homomorphism $\mathrm{C}^*(\Gamma) \prod^\omega \to M_{k_n}$, resp. $\mathrm{C}^*(\Gamma) \to \prod_\omega M_{k_n}$ lifts to a *-homomorphism.

The group is very flexibly HS, resp. operator norm, stable if for any approximate representation $\{\Gamma \to \mathcal{U}_{k_n}\}$ there is a sequence of finite-dimensional Hilbert spaces $\mathcal{H}_n$, representations $\{\pi_n : \Gamma \to \mathcal{U}(\mathcal{H}_n)\}$ and isometric inclusions $v_n : \mathbb{C}^{k_n} \to \mathcal{H}_n$ so that the approximate representations get pointwise arbitrarily close to the compressions $v_n^* \pi_n v_n$.

In [14, Definition 6.8], very flexible stability is given a lifting characterization which makes its connection to LLP (via [14, Theorem 3.1]) more clear.

[19] This is essentially a "relative property (T)" obstruction.

[20] Again, if Connes Embedding Problem were true, this would have implied this were true, which is a good exercise after the end of the lectures. Pisier has a nice aiming at this problem.

[21] [4, Chapter II.8.4] is a good resource for the details, but here's the gist. $\mathrm{Ext}(A)$ is the semi-group of

3 Lance's Tensor Characterization of WEP

To gain a greater appreciation for having a unique tensor norm on $A \odot B$, consider the following intuitive property for $\otimes_{\min}$ that can fail for $\otimes_{\max}$:

Propostion 3.1. *If $A \subseteq B$ as* C*-*algebras and C is another* C*-*algebra, then*

$$A \otimes_{\min} C \subseteq B \otimes_{\min} C.$$

Indeed, this means exactly that the norm on $A \odot C$ inherited as a subset of $B \otimes_{\min} C$ is the same as then $\|\cdot\|_{\min}$ norm on $A \odot C$, which follows from the fact that $\|\cdot\|_{\min}$ is independent of faithful representation (and a faithful representation of A can be obtained by restricting a faithful representation of B).

However, this can fail for $\otimes_{\max}$. It is perhaps more informative here to understand *why* there can be counterexamples than to see actual counterexamples. To that end, notice that we automatically have the containment $A \odot C \subseteq B \odot C$. Hence, a representation on the larger algebra restricts to a representation on the smaller algebra, but there may be representations on $A \odot C$ that do not extend to $B \odot C$. This means that, for $x \in A \odot C$,

$$\|x\|_{A \otimes_{\max} C} \geq \|x\|_{B \otimes_{\max} C}.$$

As would be expected, it turns out to be quite significant when this property does hold for $\otimes_{\max}$ for C*-algebras $A \subseteq B$, and C. It also turns out to be quite significant for this property to hold for a given C*-algebra A no matter our choice of $B \supseteq A$ and C. *This* is the value of weak expectations.

Theorem 3.2. *[32] Let B be a* C*-*algebra with* C*-*subalgebra A. Then TFAE:*

1. *A embeds relatively weakly injectively into B.*
2. *$A \otimes_{\max} C \subseteq B \otimes_{\max} C$ for any* C*-*algebra C.*
3. *$A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq B \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$*
4. *$A \otimes_{\max} \mathrm{C}^*(\mathbb{F}) \subseteq B \otimes_{\max} \mathrm{C}^*(\mathbb{F})$ for some/every non-abelian free group $\mathbb{F}$.*

For the (4)⇒(1) portion, we will invoke Lance's Tensor Product Trick from [32] (called *The Trick* in [7, Chapter 3]), so we shall give its proof first. Lance's Trick is stated for any C*-tensor product norm satisfying certain properties, which happen to be satisfied by $\|\cdot\|_{\max}$. Since we will only need it for this norm, we will state the proposition in terms of this norm. Likewise, we will stick to the unital version, though the nonunital holds just the same.

extensions of A by $\mathcal{K}$. Such an extension gives rise to a *-homomorphism $A \to B(\mathcal{H})/\mathcal{K} = \mathcal{Q}(\mathcal{H})$ called the Busby invariant. Modulo moding out by the right equivalences, $\mathrm{Ext}(A)$ is essentially the collection of *-homomorphisms $A \to B(\mathcal{H})/\mathcal{K} = \mathcal{Q}(\mathcal{H})$. On this set, we can define addition by direct sum (using the fact that $M_2(\mathcal{Q}(\mathcal{H})) \cong \mathcal{Q}(\mathcal{H})$), which makes $\mathrm{Ext}(A)$ into a semigroup. A *-homomorphism $A \to \mathcal{Q}(\mathcal{H})$ is invertible in this semigroup essentially when it has a ccp lift. Thus LLP implies $\mathrm{Ext}(A)$ is a group. Kirchberg proved in [29, Theorem 1.2] that $\mathrm{Ext}(CA)$ is a group (where $CA = C_0(0,1] \otimes A$) iff A has the LLP. (Note that if A has LLP, then CA has LLP, and so CA has LLP, so the converse is the challenge.) In [17], Enders and Shulman proved that $\mathrm{Ext}(A)$ being a group implies LLP for certain A arising as inductive limits of LLP and RFD C*-algebras, including $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$.

Lemma 3.3 (Lance's Tensor Product Trick)**.** *[32] Let $A \subseteq B$ and C be unital* C*-*algebras with $1_A = 1_B$ and such that*

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C.$$

Given representations $\pi_A : A \to B(\mathcal{H})$ and $\pi_C : C \to B(\mathcal{H})$ with commuting ranges, there exists a ucp map $\phi : B \to \pi_C(C)'$ which extends π_A.

Note that this essentially gives you just a touch more control over Arveson's extension of π_A– which is exactly how the proof goes.

Proof. By universality, $\pi_A \times \pi_C : A \odot C \to B(\mathcal{H})$ extends to a *-homomorphism $\pi_A \times \pi_C : A \otimes_{\max} C \to B(\mathcal{H})$. Since

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C,$$

$\pi_A \times \pi_C$ extends to a ucp map $\Phi : B \otimes_{\max} C \to B(\mathcal{H})$ by injectivity of $B(\mathcal{H})$. Let $\phi : B \to B(\mathcal{H})$ be given by $\phi(b) = \Phi(b \otimes 1_C)$.

To see that $\phi(B) \subseteq \pi_C(C)'$, notice that $\Phi_{\mathbb{C}1_B \otimes C} = \Phi_{\mathbb{C}1_A \otimes C} = \pi_C$ is a *-homomorphism, and so $\mathbb{C}1_B \otimes C$[22] is in the multiplicative domain[23] of Φ. Furthermore, $B \otimes \mathbb{C}1_C \subseteq (\mathbb{C}1_B \otimes C)'$. So, for $b \in B$ and $c \in C$,

$$\begin{aligned}\phi(b)\pi_C(c) &= \Phi(b \otimes 1_C)\Phi(1_B \otimes c) \\ &= \Phi((b \otimes 1_C)(1_B \otimes c)) \\ &= \Phi((1_B \otimes c)(b \otimes 1_C)) \\ &= \pi_C(c)\phi(b).\end{aligned}$$

□

Now, we are ready to give the proof of the theorem.

Proof of Theorem 3.4. That (2) ⇒ (3) is clear. For (1)⇒(2), we follow the proof for Prop 3.6.2 in [7]:

(1)⇒(2) Suppose $A \subseteq B$ relatively weakly injectively and C is any other C*-algebra. First note that, by the universal property of $\otimes_{\max}$, the embedding $A \odot C \hookrightarrow B \otimes_{\max} C$ extends to a *-homomorphism

$$\rho : A \otimes_{\max} B \to B \otimes_{\max} C.$$

We have only to show that this map is injective. To do so, we will take a faithful representation $\pi : A \otimes_{\max} C \to B(\mathcal{H})$ and show that π factors through ρ. Let $\pi_A : A \to B(\mathcal{H})$ and $\pi_C : C \to B(\mathcal{H})$ be the restrictions of π with commuting ranges guaranteed by universality. In particular, $\pi_C(C)$ and $\pi_A(A)''$ are two commuting subspaces of $B(\mathcal{H})$ and hence their natural inclusion maps $\iota_{\pi_C(C)}$ and $\iota_{\pi_A(A)''}$ are two

[22] Recall that finite dimensional C*-algebras are nuclear, so there is only one tensor product norm

[23] The multiplicative domain of Φ is $\{x \in B \otimes_{\max} C : \Phi(xy) = \Phi(x)\Phi(y)\ \&\ \Phi(yx) = \Phi(y)\Phi(x)\ \forall\, y \in B \otimes_{\max} C\}$, or the largest subspace X such that $\Phi|_X$ is a $*$-homomorphism. See Proposition A.13 for more.

$*$-homomorphisms with commuting ranges. Again, the universal property of $\otimes_{\max}$ allows us to extend $\iota_{\pi_A(A)''} \times \iota_{\pi_C(C)}$ to a *-homomorphism

$$\iota : \pi_A(A)'' \otimes_{\max} \pi_C(C) \to B(\mathcal{H}).$$

Since A has the WEP and $A \subseteq B$, there is a ccp extension $\phi : B \to \pi_A(A)''$ of π_A. Then, by functoriality of $\otimes_{\max}$, $\phi \odot \pi_C$ extends to a ccp map

$$\phi \otimes \pi_C : B \otimes_{\max} C \to \pi_A(A)'' \otimes_{\max} \pi_C(C).$$

Then, $\pi = \iota \circ (\phi \otimes \pi_C) \circ \rho$. Indeed, for $a \otimes c \in A \otimes_{\max} C$,

$$\iota \circ (\varphi \otimes \pi_C) \circ \rho(a \otimes c) = \iota(\varphi(a) \otimes \pi_C(c)) = \iota(\pi_A(a) \otimes \pi_C(c)) = \pi_A(a)\pi_C(c) = \pi(a \otimes c),$$

and likewise for sums of simple tensors. Thus ρ is injective.

(3)⇒(4) Let $\mathbb{F}$ be any free group and $t = \sum_{i=1}^n a_i \otimes x_i \in A \odot \mathrm{C}^*(\mathbb{F})$. Then, by Lemma 2.8, there is a copy of $\mathrm{C}^*(\mathbb{F}_\infty)$ in $\mathrm{C}^*(\mathbb{F})$ containing $\{x_1, ..., x_n\}$ (which we will identify with $\mathrm{C}^*(\mathbb{F}_\infty)$ for simplicity of notation), and there is a conditional expectation from $\mathrm{C}^*(\mathbb{F})$ onto this copy of $\mathrm{C}^*(\mathbb{F}_\infty)$. In particular, this is a weak expectation, and so (1) ⇒ (2) shows that, for any C^*-algebra C,

$$C \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq C \otimes_{\max} \mathrm{C}^*(\mathbb{F}).$$

Interpreting this in terms of norms and using our assumption from (3), we have

$$\|t\|_{A \otimes_{\max} \mathrm{C}^*(\mathbb{F})} = \|t\|_{A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)} = \|t\|_{B \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)} = \|t\|_{B \otimes_{\max} \mathrm{C}^*(\mathbb{F})}.$$

(4)⇒(1) Embed $\pi_u(A) \subseteq A^{**} \subseteq B(\mathcal{H}_u)$ where $(\pi_u, B(\mathcal{H}_u))$ is the universal representation for A. Let $\mathbb{F}$ be a free group on $|U(\pi_u(A)')|$ generators, and let $\pi : \mathrm{C}^*(\mathbb{F}) \to \pi_u(A)' \subseteq B(\mathcal{H}_u)$ be the surjective *-homomorphism induced by mapping generators of $\mathbb{F}$ to the unitaries of $\pi_u(A)'$. Now, we use Lance's Trick with $C = \mathrm{C}^*(\mathbb{F})$, $B = B$, $B(\mathcal{H}) = B(\mathcal{H}_u)$, $\pi_A = \pi_u$, and $\pi_C = \pi$. Then, we get a ccp map $\phi : B \to \pi_u(A)'' = A^{**}$.

□

Theorem 3.2 gives us a corresponding characterization for WEP, where we use Proposition 1.16 to reduce WEP down to checking just one tensor product embedding:

Corollary 3.4. *[32] Let A be a C^*-algebra. Then TFAE:*

1. *A has the WEP.*
2. *For any C^*-algebra B with $A \subseteq B$ and any C^*-algebra C,*

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C$$

3. *For some embedding $A \subseteq B(\mathcal{H})$*

$$A \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) \subseteq B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$$

4. For any embedding $A \subseteq B(\mathcal{H})$ and any free group $\mathbb{F}$

$$A \otimes_{\max} \mathrm{C}^*(\mathbb{F}) \subseteq B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F})$$

Just to drive home the point, we can now re-define (relative) weak injectivity/ WEP.

Definition 3.5. For a C*-algebra B, we say a C*-subalgebra $A \subseteq B$ embeds *relatively weakly injectively* if for any C*-algebra C,

$$A \otimes_{\max} C \subseteq B \otimes_{\max} C.$$

A C*-algebra A has the **Weak Expectation Property** (WEP) if every embedding into another C*-algebra is relatively weakly injective.

Remark 3.6. Fun fact: $\otimes_{\max}$ can fail to preserve embeddings, and $\otimes_{\min}$ can fail to preserve exact sequences. Another fun fact (see [7, Exercise 2.3.14] for a hint): WEP + Exact $\equiv$ Nuclear.

Corollary 3.7. *Suppose $A \subseteq B$ rwi, then for any* C*-*algebra C, if $B \otimes_{\max} C = B \otimes_{\min} C$ then $A \otimes_{\max} C = A \otimes_{\min} C$.*

Proof. Using Theorem 3.2 and the fact that $\otimes_{\min}$ does preserve embeddings, the proof reduces to the following diagram

$$\begin{array}{ccc} B \otimes_{\max} C & = & B \otimes_{\min} C \\ \cup & & \cup \\ A \otimes_{\max} C & & A \otimes_{\min} C \end{array}$$

∎

4 Kirchberg's Theorem

Here is a keystone thereom for Kirchberg's (L)LP theory.

Theorem 4.1 (Kirchberg)**.** *[30, Corollary 1.2] For any free group $\mathbb{F}$ and Hilbert space $\mathcal{H}$,*

$$\mathrm{C}^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H}) = \mathrm{C}^*(\mathbb{F}) \otimes_{\min} B(\mathcal{H}).$$

By similar arguments to those in Theorem 3.4, we can reduce the claim to the case of $\mathbb{F}_{n-1}$ for some $n \geq 3$ using Proposition 2.7 and Corollary 2.8. Likewise, we shall restrict ourselves to $\mathcal{H} = \ell^2$, where the more general case follows in similar fashion.

The ideas of the following proof is due to Pisier [43]. Our version is a mix of his original and Nate and Taka's condensed version in [7].

The crux of the proof is what is now known as Pisier's Linearization Trick (Lemma A.15), which allows us to focus our attention on the n-dimensional operator space E_n in $\mathrm{C}^*(\mathbb{F}_{n-1})$ spanned by $1 = U_0$ and the unitary generators U_k of $\mathrm{C}^*(\mathbb{F}_{n-1})$.

Remark 4.2. Since $\mathrm{C}^*(\mathbb{F}_{n-1})$ is the universal C^*-algebra generated by $n-1$ unitaries, by looking at suitable representations (on $\mathbb{C}$) we see that for any $(\alpha_k)_{k=0}^{n-1} \in \mathbb{C}^n$,

$$\left\| \sum_{k=0}^{n-1} \alpha_k U_k \right\| = \sum_{k=0}^{n-1} |\alpha_k|.$$

Hence, we see that E_n is canonically isometric to ℓ_n^1.

The following is contained in the proof of [7, Lemma 13.2.2], but it is nice to state separately because it gives a full picture of E_n (or equivalently ℓ_n^1) as an operator space.

Propostion 4.3. *This operator space is the universal operator space generated by n contractions with respect to cc maps in the sense that, given n contractions $a_0, ..., a_{n-1} \in B(\mathcal{H})$, there exists a cc map $\theta : E_n \to B(\mathcal{H})$ mapping $U_k \mapsto a_k$.*

Proof. For each a_k, we take its Halmos unitary dilation

$$v_k = \begin{bmatrix} a_n & (1 - a_n a_n^*)^{1/2} \\ (1 - a_n^* a_n)^{1/2} & -a_n^* \end{bmatrix} \in M_2(B(\mathcal{H})).$$

By universality, we have a unital $*$-homomorphism $\mathrm{C}^*(\mathbb{F}_{n-1})$ to $M_2(B(\mathcal{H}))$ given by $U_k \mapsto v_0^{-1} v_k$, which then restricts to a cc map $\phi : E_n \to M_2(B(\mathcal{H}))$. Multiplying this pointwise by v_0 and compressing to the $(1,1)$ corner $M_2(B(\mathcal{H}))$ gives us a cc map $\theta : E_n \to B(\mathcal{H})$ defined by $U_k \mapsto (v_0 \phi(U_k))_{11} = a_k$. □

As described in [7, Appendix B], by duality we get a one-to-one correspondence between elements $z = \sum_{k=0}^{n-1} U_k \otimes x_k \in E_n \otimes_{\min} B(\ell^2)$[24] and maps $T_z \in \mathrm{CB}(\ell_n^\infty, B(\ell^2))$ given by

$$T_z((\alpha_k)_{k=0}^{n-1}) = \sum \alpha_k x_k. \tag{4.1}$$

[24]The min norm for operator spaces is the one inherited from the minimal tensor product of ambient C^*-algebras.

Lemma 4.4. *Let $z = \sum_{k=0}^{n-1} U_k \otimes x_k \in E_n \otimes_{\min} B(\ell^2)$, and define $T_z : \ell_n^\infty \to B(\ell^2)$ as in (4.1). Then $\|z\|_{\min} = \|T_z\|_{cb}$ for every $z \in E_n \otimes B(\ell^2)$.*

Proof. To see that $\|z\|_{\min} \le \|T\|_{cb}$, note that z can be written as $(\mathrm{id}_{E_n} \otimes T_z)((U_k)_{k=0}^{n-1})$ where $(U_k)_{k=0}^{n-1} \in E_n \otimes \ell_n^\infty$ has norm 1. So by Proposition B.10,

$$\|z\|_{\min} \le \|\mathrm{id}_{E_n} \otimes T_z\| \le \|\mathrm{id}_{E_n} \otimes T_z\|_{cb} = \|T_z\|_{cb}.$$

On the other hand, let $z = \sum U_k \otimes x_k \in E_n \otimes_{\min} B(\ell^2)$. To show T_z is cc, we want to show that $\mathrm{id}_{M_r} \otimes T_z : M_r \otimes \ell_n^\infty \to M_r \otimes B(\ell^2)$ is contractive for all $r \in \mathbb{N}$. Now let $(a_k)_{k=0}^{n-1} \in M_r \otimes \ell_n^\infty$ be an arbitrary contraction, i.e., $\max_k \|a_k\| \le 1$.[25] Since each a_k is a contraction, universality guarantees a cc map $\theta : E_n \to B(\mathcal{H})$ given by $U_k \mapsto a_k$. By Proposition B.10, $\theta \otimes \mathrm{id}_{B(\ell^2)} : E_n \otimes B(\ell^2) \to M_r \otimes B(\ell^2)$ is also cc. Then we have

$$\|(\mathrm{id}_{M_r} \otimes T_z)((a_k)_{k=0}^{n-1})\| = \left\| \sum_{k=0}^{n-1} a_k \otimes x_k \right\| = \|(\theta \otimes \mathrm{id}_{B(\ell^2)})(z)\| \le \|z\|_{\min}.$$

Since $r \in \mathbb{N}$ and $(a_k)_{k=0}^{n-1} \in M_r \otimes \ell_n^\infty$ were arbitrary, $\|T_z\|_{cb} \le \|z\|_{\min}$. ☐

Remark 4.5. This lemma effectively states that, not only is E_n canonically isomorphic to the Banach space dual $(\ell_n^\infty)^*$, it is canonically *completely* isometrically isomorphic to the *operator space* dual $(\ell_n^\infty)^*$. See Appendix B.3 for an explanation.

Lemma 4.6. *Let $z = \sum_{k=0}^{n-1} U_k \otimes x_k \in E_n \otimes_{\min} B(\ell^2)$ with $\|z\|_{\min} = 1$. Then there exist a_k and b_k in $B(\ell^2)$ for $k = 0, ..., n-1$ such that*

1. *$a_k^* b_k = x_k$ for each k and*
2. *$\sum_{k=0}^{n-1} a_k^* a_k = 1 = \sum_{k=0}^{n-1} b_k^* b_k$.*

Proof. Let $z = \sum_{k=0}^{n-1} U_k \otimes x_k \in E_n \otimes_{\min} B(\ell^2)$ with $\|z\|_{\min} = 1$. By Lemma 4.4, the map $T_z : \ell_n^\infty \to B(\ell^2)$ from (4.1) is cc. By the Haagerup–Paulsen–Wittstock Stinespring theorem for cc maps (Theorem A.11), there exists a Hilbert space $\mathcal{H}$, a unital *-homomorphism $\pi : \ell_n^\infty \to B(\mathcal{H})$, and isometries $V, W \in B(\ell^2, \mathcal{H})$ such that $T_z(f) = V^* \pi(f) W$ for all $f \in \ell_n^\infty$. By the proof of said theorem, we may and do assume $\mathcal{H} = \ell^2$.

Let $a_k := \pi(\delta_k) V$ and $b_k := \pi(\delta_k) W$ in $B(\ell^2)$ for $k = 0, ..., n-1$. Then,

$$a_k^* b_k = V^* \pi(\delta_k) W = T_z(\delta_k) = x_k, \text{ for each } k,$$
$$\sum_{k=0}^{n-1} a_k^* a_k = \sum_{k=0}^{n-1} V^* \pi(\delta_k) V = V^* \pi(1_{\ell_n^\infty}) V = V^* V = 1,$$

and likewise $\sum_{k=0}^{n-1} b_k^* b_k = 1$. ☐

[25] Note that, by viewing ℓ_n^∞ as the diagonals in M_n, we can view elements of $M_r \otimes \ell_n^\infty$ as block diagonals of M_r matrices, or rather sequences $(a_k)_{k=0}^{n-1}$ with $\|(a_k)_{k=0}^{n-1}\| = \max_k \|a_k\|$.

Proof of Theorem 4.1. As we mentioned before, we reduce the claim to $\mathbb{F} = \mathbb{F}_{n-1}$ and $\mathcal{H} = \ell^2$. Let E_n be the n-dimensional operator space in $\mathrm{C}^*(\mathbb{F}_{n-1})$ spanned by $1 = U_0$ and the unitary generators U_k of $\mathrm{C}^*(\mathbb{F}_{n-1})$. For the unital operator space $E_n \otimes_{\min} B(\ell^2) \subseteq \mathrm{C}^*(\mathbb{F}_{n-1}) \otimes_{\min} B(\ell^2)$, consider the formal identity map[26] $E_n \otimes_{\min} B(\ell^2) \to \mathrm{C}^*(\mathbb{F}_{n-1}) \otimes_{\max} B(\ell^2)$. The set $\{U_i \otimes V \mid 0 \leq i \leq n-1, V \in \mathcal{U}(B(\ell^2))\}$ satisfies the assumptions of Pisier's Linearization Trick (Lemma A.15) for this map. So, if we show this formal identity map is cc, then it will extend uniquely to a *-homomorphism $\mathrm{C}^*(\mathbb{F}_{n-1}) \otimes_{\min} B(\ell^2) \to \mathrm{C}^*(\mathbb{F}_{n-1}) \otimes_{\max} B(\ell^2)$, which will prove the theorem.

To that end, let $z = \sum_{k=0}^{n-1} U_k \otimes x_k \in E_n \otimes_{\min} B(\ell^2)$ with $\|z\|_{\min} = 1$. By Lemma 4.6, there exist a_k and b_k in $B(\ell^2)$ for $k = 0, ..., n-1$ such that $a_k^* b_k = x_k$ for each k and $\sum_{k=0}^{n-1} a_k^* a_k = 1 = \sum_{k=0}^{n-1} b_k^* b_k$. Hence,

$$\begin{aligned}
&\left\| \sum_{k=0}^{n-1} U_k \otimes x_k \right\|_{\mathrm{C}^*(\mathbb{F}_{n-1}) \otimes_{\max} B(\ell^2)} \\
&= \left\| \sum_{k=0}^{n-1} (1 \otimes a_k)^* (U_k \otimes b_k) \right\|_{\mathrm{C}^*(\mathbb{F}_{n-1}) \otimes_{\max} B(\ell^2)} \\
&\leq \left\| \sum_{k=0}^{n-1} (1 \otimes a_k)^* (1 \otimes a_k) \right\|^{1/2}_{\mathrm{C}^*(\mathbb{F}_{n-1}) \otimes_{\max} B(\ell^2)} \left\| \sum_{k=0}^{n-1} (U_k \otimes b_k^*)(U_k \otimes b_k) \right\|^{1/2}_{\mathrm{C}^*(\mathbb{F}_{n-1}) \otimes_{\max} B(\ell^2)} \\
&= 1,
\end{aligned}$$

where the first inequality is a consequence of the Cauchy-Schwarz inequality.[27] Thus the formal identity map $E_n \otimes_{\min} B(\ell^2) \to \mathrm{C}^*(\mathbb{F}_{n-1}) \otimes_{\max} B(\ell^2)$ is contractive. Since $B(\ell^2) \simeq M_m(B(\ell^2))$ for all $m \geq 1$, the same argument above establishes that $(\mathrm{id}_{E_n} \otimes \mathrm{id}_{B(\ell^2)}) \otimes \mathrm{id}_{M_n}$ (or rather $\mathrm{id}_{E_n} \otimes \mathrm{id}_{M_n(B(\ell^2))}$) is a contraction, and hence the formal identity is cc as desired. $\square$

[26] Meaning the identity map on the underlying algebraic tensor product but as a subspace of two different spaces.

[27] Alternatively, here's a variant of a standard trick: For $a_1, ..., a_m, b_1, ..., b_m \in A$,

write $a = \begin{pmatrix} a_1 & \cdots & a_m \\ & 0 & \end{pmatrix}$ and $b = \begin{pmatrix} b_1 & \\ \vdots & 0 \\ b_m & \end{pmatrix} \in M_m(A)$.

Then $\|\sum a_i b_i\| = \|ab\| \leq \|a\|\|b\| = \|aa^*\|^{1/2}\|b^*b\|^{1/2} = \|\sum a_i a_i^*\|^{1/2}\|\sum b_i^* b_i\|^{1/2}$.

5 Kirchberg's Tensor Characterizations of WEP and LLP

A key set of (conceptual) tools in navigating Kirchberg's theory is his tensor characterizations of the WEP and LLP.

Theorem 5.1. *For any* C^**-algebra* A*,*

$$A \text{ has the WEP } \Leftrightarrow \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\max} A = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} A.$$

The same holds true if $\mathbb{F}_\infty$ *is replaced by any non-abelian free group.*

Proof. Let $A \subseteq B(\mathcal{H})$ be a faithful embedding. By Kirchberg's Theorem 4.1, $B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) = B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$. Recall that by Proposition 1.16, A has WEP iff it embeds rwi into $B(\mathcal{H})$, and so this actually characterizes WEP.

Thus we can see the equivalence of the two statements with the following diagram:

$$\begin{array}{ccc} B(\mathcal{H}) \otimes_{\max} C^*(\mathbb{F}_\infty) & = & B(\mathcal{H}) \otimes_{\min} C^*(\mathbb{F}_\infty) \\ \cup\ (\Rightarrow) & & \cup \\ A \otimes_{\max} C^*(\mathbb{F}_\infty) & \overset{(\Leftarrow)}{=} & A \otimes_{\min} C^*(\mathbb{F}_\infty) \end{array}$$

□

For the analogous characterization of LLP, we will need the Effros-Haagerup Lifting Theorem from [15, Theorem 3.1], which characterizes the Choi-Effros Lifting Theorem. We give the version from [37, Theorem 3.6].

Theorem 5.2 (Effros-Haagerup)**.** *Let* B *be a* C^**-algebra and* $J \lhd B$*. The following are equivalent*

1. *The sequence* $0 \to C \otimes_{\min} J \to C \otimes_{\min} B \to C \otimes_{\min} B/J \to 0$ *is exact for any* C^**-algebra* C*.*
2. *The sequence* $0 \to B(\ell^2) \otimes_{\min} J \to B(\ell^2) \otimes_{\min} B \to B(\ell^2) \otimes_{\min} B/J \to 0$ *is exact.*
3. $id_{B/J}$ *is locally*[28] *liftable.*

Remark 5.3. Given a short exact sequence $0 \to J \to A \to A/J \to 0$, one has $J \otimes_{\min} C \lhd A \otimes_{\min} C$. Hence proving that $0 \to J \otimes_{\min} C \to A \otimes_{\min} C \to (A/J) \otimes_{\min} C \to 0$ is exact comes down to showing $(A \otimes_{\min} C)/(J \otimes_{\min} C) \cong (A/J) \otimes_{\min} C$, i.e., that the norm $\|\cdot\|_\alpha$ induced by the embedding $(A/J) \odot C \subseteq (A \otimes_{\min} C)/(J \otimes_{\min} C)$ is $\otimes_{\min}$.

We'll sketch (3) ⇒ (1) and refer the reader to [7, Theorem C.4] for the proof of the hard part ((1) ⇒ (2)).

[28] In [15, Theorem B], with the added assumption that J is nuclear, this characterizes when $\mathrm{id}_{B/J}$ actually lifts. This also works more generally for J *approximately injective*, meaning for any finite dimensional operator subsystems $\mathcal{S} \subseteq \mathcal{T} \subseteq B(\mathcal{H})$, any cp map $\varphi : \mathcal{S} \to J$ has a cp approximate extensions $\psi : \mathcal{T} \to J$ (meaning for any ε, there is a a cp map $\psi : \mathcal{T} \to J$ with $\psi|_{\mathcal{S}} \sim_\epsilon \varphi$).

Proof. (3) ⇒ (1) Let $\|\cdot\|_\alpha$ denote the norm on $C \odot (B/J)$ inherited from the dense embedding

$$C \odot (B/J) \cong \frac{C \odot B}{C \odot J} \subseteq \frac{C \otimes_{\min} B}{C \otimes_{\min} J}.$$

It suffices to show that for any $z = \sum_{i=1}^n c_i \otimes x_i \in C \odot (B/J)$, $\|z\|_{C \otimes_{\min} (B/J)} \geq \|z\|_\alpha$. Let $S = \operatorname{span}\{1, x_i, x_i^* \mid 1 \leq i \leq n\}$ and $\varphi : S \to B$ be a local lift of $\operatorname{id}_{B/J}$. By functoriality of $\otimes_{\min}$ for operator systems B.10, $\operatorname{id}_C \otimes_{\min} \varphi : C \otimes_{\min} S \to C \otimes_{\min} B$ is a ucp map. Thus

$$\|z\|_{C \otimes_{\min} (B/J)} = \|z\|_{C \otimes_{\min} S} \geq \| \sum c_i \otimes \varphi(x_i) \|_{C \otimes_{\min} B} \geq \| \sum c_i \otimes x_i \|_\alpha = \|z\|_\alpha.$$

□

Theorem 5.4 (Kirchberg)**.** *For a unital* C**-algebra* A*, TFAE*

1. *A has the LLP.*
2. *$A \otimes_{\min} B(\ell^2) = A \otimes_{\max} B(\ell^2)$.*
3. *$A \otimes_{\min} B(\mathcal{H}) = A \otimes_{\max} B(\mathcal{H})$ for any Hilbert space $\mathcal{H}$.*

Proof. Let $\mathbb{F}$ be a free group so that $\mathrm{C}^*(\mathbb{F}) \twoheadrightarrow A$, and let $J \lhd \mathrm{C}^*(\mathbb{F})$ be the kernel of that quotient. Recall from Proposition 2.13 that A has the LLP iff $\operatorname{id}_{\mathrm{C}^*(\mathbb{F})/J}$ is locally liftable. Then the Effros-Haagerup Lifting Theorem 5.2 says the following are equivalent:

1. A has the LLP.
2. The sequence $0 \to B(\ell^2) \otimes_{\min} J \to B(\ell^2) \otimes_{\min} \mathrm{C}^*(\mathbb{F}) \to B(\ell^2) \otimes_{\min} A \to 0$ is exact.
3. For any Hilbert space $\mathcal{H}$, the sequence $0 \to B(\mathcal{H}) \otimes_{\min} J \to B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}) \to B(\mathcal{H}) \otimes_{\min} A \to 0$ is exact[29]

So, all that remains to show is $0 \to B(\mathcal{H}) \otimes_{\min} J \to B(\mathcal{H}) \otimes_{\min} B \to B(\mathcal{H}) \otimes_{\min} A \to 0$ is exact iff $A \otimes_{\min} B(\mathcal{H}) = A \otimes_{\max} B(\mathcal{H})$ for any Hilbert space $\mathcal{H}$ (which will include ℓ^2).

Fix $\mathcal{H}$. Since $\otimes_{\max} B(\mathcal{H})$ is always an exact functor (see e.g., [7, Proposition 3.7.1]), we get a short exact sequence

$$0 \to B(\mathcal{H}) \otimes_{\max} J \to B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}) \to B(\mathcal{H}) \otimes_{\max} A \to 0$$

By Kirchberg's Theorem 4.1, $B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}) = B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F})$, and since $J \lhd \mathrm{C}^*(\mathbb{F})$, it follows from Corollary 3.7 (via Remark 1.12) that $B(\mathcal{H}) \otimes_{\max} J = B(\mathcal{H}) \otimes_{\min} J$ as well. Putting these together gives us the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & B(\mathcal{H}) \otimes_{\min} J \arrow[r, hook] \arrow[d, equal] & B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}) \arrow[r, two heads] \arrow[d, equal] & \frac{B(\mathcal{H}) \otimes_{\min} \mathrm{C}^*(\mathbb{F})}{B(\mathcal{H}) \otimes_{\min} J} \arrow[r] & 0 \\
0 \arrow[r] & B(\mathcal{H}) \otimes_{\max} J \arrow[r, hook] & B(\mathcal{H}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}) \arrow[r, two heads] & B(\mathcal{H}) \otimes_{\max} A \arrow[r] & 0
\end{tikzcd}.$$

[29] I know it says "for any C*-algebra C", but clearly this claim can be squeezed in between (1) and (2) in Theorem 5.2.

So $\frac{B(\mathcal{H})\otimes_{\min}\mathrm{C}^*(\mathbb{F})}{B(\mathcal{H})\otimes_{\min}J} = B(\mathcal{H})\otimes_{\max} A$.[30] So (by Remark 5.3) the sequence $0 \to B(\mathcal{H})\otimes_{\min} J \to B(\mathcal{H})\otimes_{\min} B \to B(\mathcal{H})\otimes_{\min} A \to 0$ is exact iff $B(\mathcal{H})\otimes_{\min} A = B(\mathcal{H})\otimes_{\max} A$. □

Theorem 5.5 (Junge-Pisier)**.** *[26] For any infinite-dimensional Hilbert space $\mathcal{H}$,*

$$B(\mathcal{H})\otimes_{\max} B(\mathcal{H}) \neq B(\mathcal{H})\otimes_{\min} B(\mathcal{H}).$$

Corollary 5.6. *For any infinite-dimensional Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ does not have the LLP.*

Recall that $B(\mathcal{H})$ has the WEP, so we now know that the WEP does not imply the LLP. In fact, we now know all of the following conjectures (plus a few more) are false:

Theorem 5.7 ([29])**.** *The following conjectures are equivalent:*

1. *Ext(A) is a group for every separable unital C^*-algebra A with the WEP.*
2. *Every finite dimensional operator system is unitally completely isometrically isomorphic to an operators system in $\mathrm{C}^*(\mathbb{F}_\infty)$.*
3. *For every pair A and B of separable unital C^*-algebras with the WEP,*
$$A\otimes_{\max} B = A\otimes_{\min} B.$$
4. *The WEP and approximate injectivity are equivalent. (See Footnote 28.)*

Let's tie everything up in a nice bow.

Theorem 5.8 (Kirchberg)**.** *[29] Let A and B be C^*-algebras. Then*

1. *A has the LLP $\iff A\otimes_{\max} B(\ell^2) = A\otimes_{\min} B(\ell^2)$,*
2. *B has the WEP $\iff \mathrm{C}^*(\mathbb{F}_\infty)\otimes_{\max} B = \mathrm{C}^*(\mathbb{F}_\infty)\otimes_{\min} B$, and*
3. *A has the LLP and B has the WEP $\Rightarrow A\otimes_{\max} B = A\otimes_{\min} B$.*

Naturally, ℓ^2 can be replaced with any infinite-dimensional Hilbert space and $\mathbb{F}_\infty$ with any non-abelian free group.

Proof. It remains to show (3). To that end, faithfully embed $B \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Since B has the WEP, this embedding is realitively weakly injective ($A\otimes_{\max} B \subseteq A\otimes_{\max} B(\mathcal{H})$); since A has the LLP, $A\otimes_{\max} B(\mathcal{H}) = A\otimes_{\min} B(\mathcal{H})$. Thus we apply Corollary 3.7 to get (3). □

Propostion 5.9. *Here is a good place to remark that LLP and WEP pass to relatively weakly injective subalgebras. Indeed, this is an immediate consequence of Theorem 5.8 and Corollary 3.7.*

Since $\mathrm{id}_{A^{**}}$ is a weak conditional expectation for $A \subseteq A^{**}$, we get the following corollary:

Corollary 5.10. *If A^{**} is WEP, resp. LLP, then so is A.*

[30] Notice that this bit is just always true.

6 Main Theorem

Theorem 6.1 (Kirchberg)**.** *The following are equivalent:*

1. $\mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty)$.
2. $\mathrm{C}^*(\mathbb{F}_\infty)$ *has the WEP.*
3. $\mathrm{C}^*(\mathbb{F})$ *has WEP for every free group.*
4. *All* C^**-algebras are QWEP. (Kirchberg's QWEP Conjecture)*
5. *LLP* $\Rightarrow$ *WEP.*

Proof. First, we handle the immediate consequences of Theorem 5.8.

(1)$\Leftrightarrow$(2) This follows immediately from Theorem 5.8(2).

(4)$\Rightarrow$(1) This follows from Theorem 5.8(3) and the fact that $\mathrm{C}^*(\mathbb{F}_\infty)$ has the LLP.

(2)$\Rightarrow$(3) Since $\mathrm{C}^*(\mathbb{F})$ has the LLP for any free group $\mathbb{F}$, by Theorem 5.8(3), if $\mathrm{C}^*(\mathbb{F}_\infty)$ has the WEP, then

$$\mathrm{C}^*(\mathbb{F}) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) = \mathrm{C}^*(\mathbb{F}) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty)$$

for any free group $\mathbb{F}$. But by Theorem 5.8(2), this is equivalent to $\mathrm{C}^*(\mathbb{F})$ having the WEP.

(3)$\Rightarrow$(4) Since any unital C^*-algebra can be identified with the quotient of $\mathrm{C}^*(\mathbb{F})$ for some free group $\mathbb{F}$, we have that any unital C^*-algebra is QWEP, and the non-unital case follows from the fact for unitizations[31].

(4)$\Rightarrow$(5) Suppose A has the LLP. By assumption, there is a WEP C^*-algebra and B with two sided closed ideal J such that $A = B/J$. Since $\mathrm{id}_{B/J}$ locally lifts, the Effros-Haagerup Lifting Theorem (Theorem 5.2) tells us that the sequence $0 \to \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} J \to \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} B \to \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} A \to 0$ is exact. Since B, and hence also J, has the WEP by Theorem 5.8, we get the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} J \arrow[r, hook] \arrow[d, equal] & \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} B \arrow[r, two heads] \arrow[d, equal] & \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} A \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\max} J \arrow[r, hook] & \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\max} B \arrow[r, two heads] & \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\max} A \arrow[r] & 0 \end{tikzcd} .$$

Thus $\mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\max} A = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} A$, and so A has the WEP by Theorem 5.8.

□

[31] For example, because a non-unital C^*-algebra is an ideal (rwi subalgebra) in its unitization and Proposition 7.2

Remark 6.2. Now, sometimes we prefer to work instead with other free groups– like $\mathrm{C}^*(\mathbb{F}_2)$. So we remark here that $\mathrm{C}^*(\mathbb{F}_\infty)$ can be replaced with any non-abelian free group– or heck, any group in which $\mathbb{F}_\infty$ embeds. This follows from Proposition 5.9.

With that in mind, let's just quickly touch base on my favorite version of the theorem.

Theorem 6.3. *All of the statements in Theorem 6.1 are equivalent to* $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ *being RFD.*

The same holds for any discrete group with a non-abelian free subgroup whose full group C^**-algebra is RFD and has LLP, like* $\mathbb{Z}_3 * \mathbb{Z}_3$, *or* $\mathbb{Z}_m^{*n}$ *where at least one of* m *and* n *is larger than* 2.

Proof. Note that for any discrete groups Γ, Λ, we have $\mathrm{C}^*(\Gamma \times \Lambda) \cong \mathrm{C}^*(\Gamma) \otimes_{\max} \mathrm{C}^*(\Lambda)$ by comparing universal properties[32].

Next note that for any RFD C^*-algebras A and B, we have $A \otimes_{\max} B$ is RFD iff $A \otimes_{\max} B = A \otimes_{\min} B$. Indeed, this is because $A \otimes_{\min} B$ is always RFD (seen by taking tensor products of separating families of finite dimensional representations) and because any finite-dimensional representation of $A \otimes_{\max} B$ factors through $A \otimes_{\min} B$. (See Proposition C.3 for a full proof).

Thus for $\mathrm{C}^*(\mathbb{F}_2)$ (which is RFD by [9]), it follows that $\mathrm{C}^*(\mathbb{F}_2 \times \mathbb{F}_2)$ is RFD iff $\mathrm{C}^*(\mathbb{F}_2) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_2) = \mathrm{C}^*(\mathbb{F}_2) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_2)$ iff $\mathrm{C}^*(\mathbb{F}_2)$ is WEP (Theorem 5.8) iff $\mathrm{C}^*(\mathbb{F}_\infty)$ is WEP (Proposition 5.9).

Now, suppose Γ is an RFD discrete group with LLP and a subgroup isomorphic to $\mathbb{F}_2$. Then by Proposition C.3, $\mathrm{C}^*(\Gamma \times \Gamma) \cong \mathrm{C}^*(\Gamma) \otimes_{\max} \mathrm{C}^*(\Gamma)$ is RFD iff $\mathrm{C}^*(\Gamma) \otimes_{\max} \mathrm{C}^*(\Gamma) \cong \mathrm{C}^*(\Gamma) \otimes_{\min} \mathrm{C}^*(\Gamma)$. Since $\mathbb{F}_2 \subseteq \Gamma$, $\mathrm{C}^*(\mathbb{F}_2) \overset{\subseteq}{\mathrm{rwi}} \mathrm{C}^*(\Gamma)$, and so by Corollary 3.7, $\mathrm{C}^*(\mathbb{F}_2) \otimes_{\max} \mathrm{C}^*(\Gamma) \cong \mathrm{C}^*(\mathbb{F}_2) \otimes_{\min} \mathrm{C}^*(\Gamma)$, which by Kirchberg's Theorem 5.8, implies $\mathrm{C}^*(\Gamma)$ has WEP. Since WEP passes to rwi subalgebras, it follows that $\mathrm{C}^*(\mathbb{F}_2)$, and correspondingly $\mathrm{C}^*(\mathbb{F}_\infty)$ (which sits rwi inside $\mathrm{C}^*(\mathbb{F}_2)$) has WEP.

On the other hand, if $\mathrm{C}^*(\mathbb{F}_\infty)$ has WEP, then, since $\mathrm{C}^*(\Gamma)$ has LLP, $\mathrm{C}^*(\mathbb{F}_2) \otimes_{\max} \mathrm{C}^*(\Gamma) \cong \mathrm{C}^*(\mathbb{F}_2) \otimes_{\min} \mathrm{C}^*(\Gamma)$ by Kirchberg's Theorem 5.8. This in turn implies $\mathrm{C}^*(\Gamma)$ has WEP, which, again using Kirchberg's Theorem 5.8, implies $\mathrm{C}^*(\Gamma) \otimes_{\max} \mathrm{C}^*(\Gamma) \cong \mathrm{C}^*(\Gamma) \otimes_{\min} \mathrm{C}^*(\Gamma)$. Hence $\mathrm{C}^*(\Gamma \times \Gamma)$ is RFD.

For why this applies to free products of cyclic groups, we already know LLP is preserved by full free products (Pisier); the fact that $\mathrm{C}^*(\mathbb{Z}_m^{*n})$ is RFD follows from the fact due to Excel and Loring that full free products of RFD C^*-algebras are RFD ([18]); and the fact that $\mathbb{Z}_m^{*n}$ (with $(m, n) \neq (2, 2)$) contains a copy of $\mathbb{F}_2$ is a consequence of the Ping-Pong Lemma (see e.g. [25, Lemma D.2] for a proof). ☐

Exercise 6.4. In fact, $\mathbb{Z}_m^{*n}$ (with $(m, n) \neq (2, 2)$) could replace $\mathbb{F}_\infty$ in the statement of Theorem 6.1.

[32]Meaning each are universal C^*-algebras generated by two sets of commuting unitaries satisfying the defining relations of Γ and Λ respectively.

7 Connections to CEP

Connes' Embedding Problem was an unnumbered remark buried in the proof of [10, Theorem 5.1]. At this stage, he is dealing with a certain von Neumann algebra N (satisfying [10, Theorem 5.1 (3)] if you must know), for which he says

> *We now construct an approximate embedding of N in $\mathcal{R}$. Apparently such an imbedding ought to exist for all* II_1*, factors because it does for the regular representation of free groups.*

Since approximate embeddings into $\mathcal{R}$ translate to honest embeddings on $\mathcal{R}^\omega$, we get the common formulation of the problem:

Question (Connes Embedding Problem)**.** *Does every separably acting type* II_1*-factor embed into some ultrapower $\mathcal{R}^\omega$ of the hyperfinite* II_1*-factor $\mathcal{R}$?*

Given Theorem 6.1, to connect these different statements to Connes Embedding Problem, we only need to show the following:

Theorem 7.1 (Kirchberg)**.** *The following statements are equivalent:*

1. *Connes Embedding Problem has an affirmative answer.*
2. *Every* C^**-algebra is QWEP.*
3. $\mathrm{C}^*(\mathbb{F})$ *has WEP for some/every non-abelian free group* $\mathbb{F}$.

What remains is (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1), the latter of which are already equipped for:

Proof of $\mathrm{C}^*(\mathbb{F}_\infty)$ *WEP* $\Rightarrow$ *CEP.* Suppose M is a II_1-factor with separable predual M_* and faithful tracial state τ. That means M is weak*-separable, and so (since M_1 is weak*-compact metrizable) M has a weak*-dense unital C^*-subalgebra which thus is generated by a countable family of unitaries.

Thus means there exists a $*$-homomorphism $\psi : \mathrm{C}^*(\mathbb{F}_\infty) \to M$ whose image is wk*-dense in M. Then $\tau\psi$ is a tracial state on $\mathrm{C}^*(\mathbb{F}_\infty)$, and from uniqueness of GNS representations, we can identify $M \simeq \pi_{\tau\psi}(\mathrm{C}^*(\mathbb{F}_\infty))''$.

If $\mathrm{C}^*(\mathbb{F}_\infty)$ has the WEP, then $\tau\psi$ is amenable by Corollary 1.22, which by Theorem 1.21 means that there exists a trace-preserving embedding $M \cong \pi_{\tau\psi}(\mathrm{C}^*(\mathbb{F}_\infty))'' \subset \mathcal{R}^\omega$.

□

7.1 CEP implies QWEP

We begin by collecting and prove/reference some relevant permanence properties for (Q)WEP. For more, see [29]. The most utilized permanence property is the following:

Propostion 7.2. *Suppose $A \subseteq C$ are* C^**-algebras, C is (Q)WEP, and there exists a weak conditional expectation from C to A^{**}. Then A is (Q)WEP.*

This proof is essentially the one in [44, Proposition 9.64].

Proof. We have already seen the claim for WEP in Proposition 5.9. For (Q)WEP, suppose B is a C*-algebra with WEP and 2-sided closed ideal J and suppose $A \subset B/J$ relatively weakly injectively. Let $D = q^{-1}(A)$ such that $J = q^{-1}(\{0\}) \subseteq D \subseteq B$ and $A = D/J$. Let C be any other C*-algebra and $\phi : C \otimes_{\max} D \to C \otimes_{\max} B$ the natural map (existing by universality of $\otimes_{\max}$). Since $C \otimes_{\max}$ is exact (i.e., preserves exact sequences, see e.g., [7, Proposition 3.7.1]), we get the following commutative diagram with exact rows.

$$\begin{tikzcd} 0 \arrow[r] & C \otimes_{\max} J \arrow[r, hook] \arrow[d, equal] & C \otimes_{\max} D \arrow[r, two heads] \arrow[d, "\phi"] & C \otimes_{\max} A \arrow[r] \arrow[d, hook, "\mathrm{rwi}"] & 0 \\ 0 \arrow[r] & C \otimes_{\max} J \arrow[r, hook] & C \otimes_{\max} B \arrow[r, two heads] & C \otimes_{\max} B/J \arrow[r] & 0 \end{tikzcd} .$$

Then the four lemma tells us ϕ is injective. Hence $D \subseteq B$ is relatively weakly injective, which means D inherits the WEP by Proposition 5.9, and so A is QWEP. □

Since $\mathrm{id}_{A^{**}}$ is a weak conditional expectation for $A \subseteq A^{**}$, we have the following.

Corollary 7.3. *A* C*-*algebra A is QWEP if A^{**} is QWEP.*[33]

We conclude the section with a few properties that we will cite and not prove.

Propostion 7.4. *Let A_i below be* C*-*algebras*

1. *A_i is (Q)WEP for every $i \in \mathcal{I}$ iff $\prod_{i \in \mathcal{I}} A_i$ is (Q)WEP.*[34]
2. *Let $\{A_i\}_{i \in \mathcal{I}}$ be an increasing net of* C*-*subalgebras in $B(\mathcal{H})$. If A_i is (Q)WEP for every $i \in \mathcal{I}$, then so is $(\bigcup A_i)''$.*[35]

With these in hand, we are ready to prove CEP implies WEP.

Recall from Example 1.20 that $\mathcal{R}^\omega$ is QWEP. Now assume CEP is true, i.e., that every separably acting type II_1-factor embeds tracially into $\mathcal{R}^\omega$.[36]

Since these are tracial von Neumann algebras (see Appendix D), it turns out this embedding is relatively weakly injective. In fact (as one can find in e.g., [7, 1.5.11])

Propostion 7.5. *If (N, τ) is a tracial von Neumann algebra, then any von Neumann subalgebra $M \subset N$ is the range of a conditional expectation $N \to M$.*

[33] This is actually an iff by Proposition 7.4 (2).

[34] One direction follows from Proposition 7.2, my favorite proof for the other Pisier's in [44]. One argues first for WEP using the characterization that a C*-algebra has WEP iff $A \otimes_{\min} \mathrm{C}^*(\mathbb{F}) = A \otimes_{\max} \mathrm{C}^*(\mathbb{F})$. For each $t \in A \odot \mathrm{C}^*(\mathbb{F})$,

$$\|t\|_{\prod A_i \otimes_{\min} \mathrm{C}^*(\mathbb{F})} = \sup_i \|t\|_{A_i \otimes_{\min} \mathrm{C}^*(\mathbb{F})} = \sup_i \|t\|_{A_i \otimes_{\max} \mathrm{C}^*(\mathbb{F})} = \|t\|_{\prod A_i \otimes_{\max} \mathrm{C}^*(\mathbb{F})}.$$

Then the fact for QWEP follows immediately.

[35] This one requires extra properties about QWEP no matter how you slice it. You can choose to go Ozawa's route (see [37, Proposition 4.1] or likewise [7, 13.3.6]) or Pisier's Route [44, Proposition 9.74].

[36] Just like everyone else in the literature, I am going to get sloppy about saying "some ultrapower".

Thus CEP implies that every separably acting II_1-factor has QWEP. The plan is to bootstrap our way up from here to every C*-algebra. First we'll show it holds for all von Neumann algebras.

The first step is from separably acting II_1-factors to separably acting tracial von Neumann algebra. In [45, Theorem 4.1], Popa proved that the free product of any tracial von Neumann algebra (M, τ) with the free factor $L(\mathbb{F}_2)$ is a factor (following from the fact that $L(\mathbb{F}_2)' \cap (M * L(\mathbb{F}_2))$ is trivial). In particular, any finite von Neumann algebra (M, τ) embeds in a trace preserving way into a finite factor, which is separably acting provided that M is. Hence we have the following.

Theorem 7.6 (Popa)**.** *If every* II_1*-factor with separable predual embeds into some ultrapower* $\mathcal{R}^\omega$ *of the hyperfinite* II_1*-factor* $\mathcal{R}$*, then so does every tracial von Neumann algebra with separable predual.*

Thus every separably acting tracial von Neumann algebra inherits QWEP from $\mathcal{R}^\omega$. By Proposition D.5, any semifinite von Neumann algebra can be written as $\overline{\bigcup_i M_i}''$ where $\{M_i\}_i$ is an increasing net of finite (tracial) unital von Neumann sub-algebras. Hence by Proposition 7.4, it follows that all semifinite von Neumann algebras are QWEP.

To cover all over von Neumann algebras (particularly those of type III), it suffices to show that any von Neumann algebra embeds relatively weakly injectively into a semifinite one. That fact follows from Takesaki's Modular Theory: for any von Neumann algebra M, the crossed produce $M \rtimes_\alpha \mathbb{R}$ by the modular flow is semifinite, and as a crossed product, there is a conditional expectation onto the copy of M inside $M \rtimes_\alpha \mathbb{R}$.[37] It follows that all von Neumann algebras are QWEP.

The fact that all C*-algebras now follows from Corollary 7.3 (and Sakai's Theorem).

Thus CEP $\Rightarrow$ QWEP.

[37] This is often presented as a corollary to Takesaki's Duality Theorem [47, Theorem X.2.3] ([48, Theorem 4.5]) combined with the fact ([47, Theorem XI.1.1]) that any type III von Neumann algebra (the only type not covered by the semifinite case) can be written as a crossed product $N \rtimes_\theta \mathbb{R}$ where N is semifinite (the "dual" to this embedding gives us the desired embedding $M \hookrightarrow N \rtimes \mathbb{R}$).

In fact, because QWEP is established for all semifinite von Neumann algebras at this point and since QWEP is preserved under finite direct sums by the type decomposition of von Neumann algebras (see [46, Theorem V.1.19] or [7, Theorem 1.3.4]), it only remains to show QWEP for type III. But for ease of reading, we ignore mention of type III.

8 Epilogue: Tsirelson's problem

Connes' Embedding problem was refuted in [24], which actually gave a direct refutation of Tsirelson's Problem. Here's a very rough sketch of the connection.

Here's a setup borrowed from (and likely described better in) [25]: In Quantum Mechanics, a measurement with m possible outcomes can be described by a projection valued measure (PVM), i.e., a collection of m projections $\{E_1, ..., E_m\}$ acting on some Hilbert space $\mathcal{H}$ such that $\sum_i P_i = 1$.

A physical state is mathematically represented by a state φ on $B(\mathcal{H})$ (or dually with a unit vector $\varphi \in \mathcal{H}$). Let's say we are performing measurements on n properties[38]. (For simplicity, still each of these n measurements gets m possible outcomes). So we get n sets of PVMs $\{\{E_{x,a}\}_{1\leq a\leq m}\}_{1\leq x\leq n}$. The probability of obtaining an outcome a when measuring property x on a quantum system in the state φ is given by the expression

$$p(a \mid x) = \varphi(E_{x,a}).$$

Now, let's say there are two independent labs A and B, each running n experiments, each of which has m possible outcomes. (Again, we make the numbers match for simplicity.) We want to determine the joint conditional probabilities of lab A getting outcome a, resp. b, when measuring property x, resp. y. The question is how does one model this.

The collection of these associated probability distributions are called correlations. Formally, a *correlation* is just a tuple

$$p = \{p(a, b \mid x, y)\}_{1\leq x,y\leq n}^{1\leq a,b\leq m} \in [0, 1]^{m^2n^2}$$

such that $\sum_{1\leq a,b\leq m} p(a, b \mid x, y) = 1$, i.e., for fixed x, y it's a probability distribution.

It is a *quantum (spatial) correlation* if for some finite-dimensional (arbitrary) Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, unit vector $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B$, and PVMs $\{E_{x,a}\} \subseteq B(\mathcal{H}_A), \{F_{y,b}\} \subseteq B(\mathcal{H}_B)$

$$p(a, b \mid x, y) = \langle (E_{x,a} \otimes F_{y,b})\xi, \xi\rangle, \ \forall x, y, a, b.$$

It is a *quantum commuting correlation* if for some Hilbert space $\mathcal{H}$, unit vector $\xi \in \mathcal{H}$ and PVMs $\{E_{x,a}\} \in \{F_{y,b}\}' \subset B(\mathcal{H})$

$$p(a, b \mid x, y) = \langle E_{x,a}F_{y,b}\xi, \xi\rangle, \ \forall x, y, a, b.$$

There are more types of correlations, but for the sake of staying focused, here are the ones we care about. So we arrive at our correlation sets.

$$\begin{aligned}
C_q(m, n) &= \{\text{all possible quantum correlations for n experiments with m outcomes}\}\\
C_{qs}(m, n) &= \{\text{all possible quantum spatial correlations for [same]}\}\\
C_{qc}(m, n) &= \{\text{all possible quantum commuting correlations for [same]}\}
\end{aligned}$$

Remark 8.1. It is well known in quantum information that $\overline{C_q(m, n)} = C_{qs}(m, n)$, a fact which comes down to residual finite dimensionality of spatial tensor products. Maybe you can get a feel for how that features after the discussion below.

[38] Likely the spin of some particle in a given direction

Question (Tsirelson)**.** *Is* $\overline{C_q(m,n)} = C_{qc}(m,n)$ *for all* $m,n \geq 2$ *(*$(m,n) \neq (2,2)$*)*[39]*? (Equivalently, is* $C_{qs}(m,n) = C_{qc}(m,n)$ *for all* $m,n \geq 2$ *(*$(m,n) \neq (2,2)$*)?)*

In [24], Ji, Natarajan, Vidick, Wright, and Yuen disprove this conjecture. In fact, they exhibit an explicit non-local game (a nice theoretical model for experiments) which yields an explicit correlation that is in the set $C_{qc}(m,n)$ but not in $C_{qa}(m,n)$.[40]

So what does that have to do with our topic?

Theorem 8.2 (Fritz, Junge–Navascues–Palazuelos–Perez-Garcia–Scholz–Werner, Ozawa)**.** *Tsirelson's problem has a positive solution iff* $\mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty) = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty)$.

Fritz ([20]) and Junge–Navascues–Palazuelos–Perez-Garcia–Scholz–Werner ([25]) show that a positive solution to Kirchberg's conjecture implies that Tsirelson's problem has a positive solution. (Ozawa proved the converse, which is a fair bit more involved. Pisier gives a good account in [44].)

The papers [20] and [25] contain many of the same core ideas and the results were posted to arXiv on the same day. I personally find Tobias Fritz's version [20] more accessible, so that's what I draw from here.

Theorem 8.3 (Fritz)**.** *[20, Proposition 3.4] For* $m,n \geq 2$,

$$\overline{C_q(m,n)} = \{\{\rho(E_{x,a} \otimes E_{y,b})\}_{1\leq x,y\leq n}^{1\leq a,b\leq m} \mid \rho \in S(\mathrm{C}^*(\mathbb{Z}_m^{*n}) \otimes_{\min} \mathrm{C}^*(\mathbb{Z}_m^{*n}))\}$$

i.e., $p \in \overline{C_q(m,n)}$ *iff there exists a state* $\rho \in S(\mathrm{C}^*(\mathbb{Z}_m^{*n}) \otimes_{\min} \mathrm{C}^*(\mathbb{Z}_m^{*n}))$ *so that* $p(a,b \mid x,y) = \rho(E_{x,a} \otimes E_{y,b})$ *for all* a,b,x,y*. Likewise,*

$$C_{qc}(m,n) = \{\{\rho(E_{x,a} \otimes E_{y,b})\}_{1\leq x,y\leq n}^{1\leq a,b\leq m} \mid \rho \in S\big(\mathrm{C}^*(\mathbb{Z}_m^{*n}) \otimes_{\max} \mathrm{C}^*(\mathbb{Z}_m^{*n})\big)\}$$

i.e., $p \in C_{qc}(m,n)$ *iff there exists a state* $\rho \in S\big(\mathrm{C}^*(\mathbb{Z}_m^{*n}) \otimes_{\max} \mathrm{C}^*(\mathbb{Z}_m^{*n})\big)$ *so that* $p(a,b \mid x,y) = \rho(E_{x,a} \otimes E_{y,b})$ *for all* a,b,x,y*.*

Now if $\mathrm{C}^*(\mathbb{F}_2)$ has WEP, then $\mathrm{C}^*(\mathbb{Z}_m^{*n})) \otimes_{\max} \mathrm{C}^*(\mathbb{Z}_m^{*n})) = \mathrm{C}^*(\mathbb{Z}_m^{*n})) \otimes_{\min} \mathrm{C}^*(\mathbb{Z}_m^{*n}))$ for all m,n, and so Tsirelson's conjecture would be true. [41]

[39] $m = 2 = n$ is modeled by the famous CHSH scenario, which is evidently too simple (see [20, Remark 3.6])

[40] As remarked in the paper, though the exact m and n needed for this to work are presumably quite large, and an upper bound for the required size is not known. Still, "there is no step in the proof that requires it to be astronomical; e.g. [they] believe (without proof) that 10^20 is a clear upper bound."

[41] Since this argument was not worked out explicitly in Theorem 6.3, note that by Theorem 6.3, Kirchbergs conjectures are equivalent to saying $\mathrm{C}^*(\mathbb{Z}_m^{*n}) \times \mathbb{Z}_m^{*n})) = \mathrm{C}^*(\mathbb{Z}_m^{*n})) \otimes_{\max} \mathrm{C}^*(\mathbb{Z}_m^{*n}))$ is RFD for all $m,n \geq 2$ with $(m,n) \neq (2,2)$ (in which case the group is amenable). Since $\mathrm{C}^*(\mathbb{Z}_m^{*n}))$ is RFD ([18]), by Proposition C.3, this is equivalent to saying $\mathrm{C}^*(\mathbb{Z}_m^{*n})) \otimes_{\max} \mathrm{C}^*(\mathbb{Z}_m^{*n})) = \mathrm{C}^*(\mathbb{Z}_m^{*n})) \otimes_{\min} \mathrm{C}^*(\mathbb{Z}_m^{*n}))$.

A Operator Systems, Operator Spaces, and their maps

As one dives into the details of Kirchberg's theory, one realizes how crucial of a role operator systems and spaces play. Here we give some key theorems, definitions, and concepts that play a role. Most if not all of the proofs are outsourced.

Definition A.1. An *operator system* $\mathcal{S}$ is a self-adjoint subspace of a unital C*-algebra A which contains the unit of A.[42] An *operator space* X is a closed subspace of a C*-algebra B.[43]

The defining structure of an operator system is its cone of positive elements (and those of its matrix amplifications $M_n(\mathcal{S})$), which it inherits from its ambient C*-algebra A. The defining structure of an operator space is its norm (and those of its matrix amplifications $M_n(X)$), which it inherits from its ambient C*-algebra B. Both of these notions, like C*-algebras, have abstract counterparts. But thanks to the Choi-Effros Theorem for operator systems and Ruan's Theorem for operator spaces, these abstract counterparts can always be concretely represented as operators on a Hilbert space (see [40] for more).

A.1 Maps on Operator Systems/Spaces

Given a linear map $\varphi : \mathcal{S} \to \mathcal{T}$ between two operator systems, resp. spaces, we define its matrix amplification $\varphi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ to be

$$\varphi^{(n)}([x_{ij}]_{ij}^n) = [\varphi(x_{ij})]_{ij}^n.$$

Definition A.2. Given two operator systems $\mathcal{S}$ and $\mathcal{T}$ a linear map $\varphi : \mathcal{S} \to \mathcal{T}$ is *positive* if it maps positive elements to positive elements. It is *completely positive* (cp) if $\varphi^{(n)}$ is positive for all $n \in \mathbb{N}$.[44] It is *unital* if it maps the unit to the unit; a unital completely positive map is called *ucp*. It is *completely bounded* if

$$\|\varphi\|_{\mathrm{cb}} := \sup_n \|\varphi_n\| < \infty.$$

It is *contractive* if $\|\varphi\| \leq 1$ and *completely contractive* (cc) if $\|\varphi\|_{cb} \leq 1$. It is *completely isometric* (ci) if each $\varphi^{(n)}$ is isometric. It is a *complete order embedding* if it has a cp inverse and a *complete order isomorphism* if it is a surjective complete order embedding.

The notions of completely bounded, (completely) contractive, and completely isometric are defined in exactly the same way for operator spaces.

Remark A.3. Some key basic facts that can be found in [40, Chapters 2 & 3] for cp maps $\varphi : \mathcal{S} \to \mathcal{T}$ between operator systems

1. If φ is positive, then $\varphi(x^*) = \varphi(x)^*$ for all $x \in \mathcal{S}$.

[42] They are not usually assumed to be closed. There exists a now standard non-unital version due to Werner, but this will not play a role here.

[43] They are not usually assumed to be unital or self-adjoint.

[44] This definition absolutely works in the non-unital setting, e.g., for non-unital C*-algebras. Though for an operator system without a unit, there might not always be positive elements to speak of.

2. If φ is cp, then $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$ (Called the Schwarz Inequality)

3. If φ is completely positive, then it is cb with $\|\varphi(1)\| = \|\varphi\| = \|\varphi\|_{\mathrm{cb}}$. In particular, if φ is contractive (e.g., when it is unital), then it is cc. Hence completely positive contractive maps are abbreviated cpc (or ccp depending on where you put the "contractive" adjective).

Remark A.4. If a complete order embedding is unital (or more generally is contractive with contractive inverse) then by (2) above, it is also completely isometric. On the other hand, if a unital *-linear map is isometric, then it is positive by the following fact from functional calculus[45]: For a unital C*-algebra A with $a \in A$ self-adjoint,

$$a \geq 0 \iff \|\|a\|1_A - a\| \leq \|a\|.$$

It follows that a unital *-linear map $\varphi : \mathcal{S} \to \mathcal{T}$ is a complete order embedding iff it is completely isometric.

Since the category of operator systems has ucp maps as morphisms and since complete isometries are the categorical isomorphisms for the category of operator spaces, one usually just sticks with talking about unital complete isomorphisms.

Note that without unital maps (resp. for non-unital C*-algebras), these claims can fail. See for example, [11, Example 1.3].

Now for the biggest theorems for these maps: Arveson and Stinespring.

A.2 Arveson's Extension Theorem

The following is Arveson's extension theorem, which is often aptly described as an infinite-dimensional Hahn-Banach Theorem.

Theorem A.5 (Arveson's Extension Theorem). *Let A be a unital* C**-algebra with operator subsystem $\mathcal{S}$, and let $\mathcal{H}$ be any Hilbert space. Then any cpc map $\varphi : \mathcal{S} \to B(\mathcal{H})$ extends to a cpc map $\tilde{\varphi} : A \to B(\mathcal{H})$ so that $\tilde{\varphi}|_{\mathcal{S}} = \varphi$.*

As a corollary, Arveson also shows the following (presumably using something along the lines of [40, Proposition 3.5]):

Corollary A.6 (Arveson–Wittstock). *Let A be a untial* C**-algebra with unital subspace X and $\varphi : X \to B(\mathcal{H})$ a unital complete contraction. Then there exists a ucp map $\tilde{\varphi} : A \to B(\mathcal{H})$ extending φ.*[46]

Remark A.7. Arveson's theorem has a number of other slight modifications that we wont use here. For example, one can do the same by replacing the operator subsystem with a non-unital C*-subalgebra (see the discussion in [12] after Theorem 2.7). It also holds when one

[45] Although the operator systems do not enjoy the functional calculus, we move to their ambient C*-algebras, from which they borrow their positive cones, to employ it.

[46] In case you've seen the version of this theorem stating that a cc map extends to a cc map, note that in this case ϕ is a unital and cc map on a C*-algebra. Hence it is automatically ucp. (This is essentially coming from the argument for why a unital contractive linear functional on a C*-algebra is a state. See [40, Prop 2.11]).

replaces the C*-algebra A with an operator system $\mathcal{T}$. By Wittstock's extension theorem (see [40, Theorem 8.2]), one can extend cb maps out of subspaces of unital C*-algebras to cb maps on the C*-algebras.

Remark A.8. Arveson's extension theorem (and its derivatives) plus Wittstock's dilation theorem tell us that $B(\mathcal{H})$ is in fact injective in the following categories:

- Operator spaces with cb maps.
- Operator spaces with cc maps.
- Operator systems with cp maps.
- Operator systems with ucp maps.

See [40, Chapter 15] for more.

A.3 Stinespring's Dilation Theorem

The following is Stinespring's Dilation Theorem, a generalization of the Gelfan-Naimark-Segal (GNS) construction for cp maps.

Theorem A.9 (Stinespring's Dilation Theorem)**.** *Let A be a unital* C**-algebra and $\varphi : A \to B(\mathcal{H})$ a cp map. Then there exists a Hilbert space $\hat{\mathcal{H}}$, a *-homomorphism $\pi : A \to B(\hat{\mathcal{H}})$, and an operator $V : \mathcal{H} \to \hat{\mathcal{H}}$ with $\|V^*V\| = \|\varphi\| = \|\varphi(1)\|$ so that for all $a \in A$*

$$\varphi(a) = V^*\pi(a)V.$$

When φ is unital, we can identify $\mathcal{H}$ as a subspace of $\hat{\mathcal{H}}$ and replace V in the above with the projection $P_{\mathcal{H}} : \hat{\mathcal{H}} \to \mathcal{H}$.

Moreover, there is a unique (up to unitary equivalence) such dilation which is minimal in the sense that $\pi(A)V\mathcal{H}$ is dense in $\hat{\mathcal{H}}$.

Remark A.10. One can do the same for non-unital C*-algebras by unitizing the map (as in [7, Proposition 2.2.1]).

We will also want a generalization, which in its present form is best attributed to a combination of results of Wittstock, Paulse, and Haagerup.

Theorem A.11 (Haagerup, Paulsen, Wittstock)**.** *Let $X \subseteq A$ be an operator subspace of a* C**-algebra and $\phi : X \to B(\mathcal{H})$ a cc map. Then there exists a Hilbert space K, a unital *-homomorphism $\pi : A \to B(K)$, and isometries $V, W : H \to K$ such that*

$$\phi(x) = V^*\pi(x)W$$

for every $x \in X$. In particular, ϕ extends to a cc map on A.

Remark A.12. As stated in [7, Theorem B.7] and in [40, Theorem 8.4], the *-homomorphism π in the above theorem is not stated as unital. However, the Wittstock dilation on which this is based ([40, Theorem 8.3]) guarantees that π is unital, as is implicitly, resp. explicitly, stated in the proofs of [7, Theorem B.7], resp. [40, Theorem 8.4].

There is also a cb version of this theorem, see [40, Theorem 8.4].

For a cpc map $\varphi : A \to B$ between C*-algebras, an important feature is its *multiplicative domain*

$$A_\varphi = \{a \in A \mid \varphi(ab) = \varphi(a)\varphi(b), \varphi(ba) = \varphi(b)\varphi(a),\ \forall b \in A\}, \tag{A.1}$$

which is the largest C*-subalgebra of A for which φ is multiplicative. Using Stinespring's theorem, one can describe it as follows (see e.g. [7, Proposition 1.5.7] for a proof):

Propostion A.13. *Let A and B be C*-algebras and $\varphi : A \to B$ cpc. Then its multiplicative domain A_φ (A.3) is the largest C*-subalgebra for which φ restricts to a *-homomorphism, and*

$$A_\varphi = \{a \in A \mid \varphi(a^*a) = \varphi(a)^*\varphi(a) \text{ and } \varphi(aa^*) = \varphi(a)\varphi(a)^*\}.$$

Remark A.14. If $\|a\| = 1$ and $\varphi(a)$ is a unitary in B, then $a \in A_\varphi$. Indeed, using the fact that 1_B is the maximal positive element in the unit ball of B along with the Schwarz inequality, one obtains

$$\varphi(a)^*\varphi(a) = 1 \geq \varphi(a^*a) \geq \varphi(a)^*\varphi(a).$$

In other words, $\varphi(a^*a) - \varphi(a)^*\varphi(a) \geq 0$ and $\varphi(a^*a) - \varphi(a)^*\varphi(a) \leq 0$. Thus they are equal. If A has a unit, then such an element $a \in A$ forces the map to be unital since $1_B \geq \varphi(1_A) \geq \varphi(a^*a) = \varphi(a)^*\varphi(a) \geq 1_B$.

Multiplicative domains are surprisingly useful. Here's an example, which is used in a key theorem in these notes. This is (a version of) *Pisier's Linearization Trick* [42, Prop 1.7], which Haagerup and Thorbjørnsen modify to get their own Linearization Trick in [21, Section 2]. To represent both Pisier's original version and that appearing in [7, Lemma 13.2.3][47], I have adjusted the statement.

Lemma A.15 (Pisier's Linearization Trick)**.** *Let $X \subseteq B(\mathcal{H})$ a be unital operator subspace and let $\phi : X \to B(\mathcal{H}')$ be a unital complete contraction. Suppose there is a collection of elements $\{a_i\}_{i\in I}$ in X such that*

1. *$\|a_i\| = 1$ for all $i \in I$,*
2. *$\phi(a_i)$ is a unitary for all $i \in I$, and*
3. *The C*-subalgebra of $B(\mathcal{H})$ generated by X, $\mathrm{C}^*(X)$ is generated by $\{a_i\}_{i\in I}$ as a C*-algebra (i.e., $\mathrm{C}^*(X) = \mathrm{C}^*(\{a_i\}_{i\in I})$).*

Then ϕ uniquely extends to a $*-$homomorphism from $\mathrm{C}^*(X)$ to $B(\mathcal{H}')$.

Proof. By the Arveson–Wittstock Extension Theorem (Arveson's Extension Theorem for unital operator spaces– see Corollary A.6), ϕ extends to a ucp map from $\mathrm{C}^*(X)$ to $B(\mathcal{H}')$, also denoted ϕ.

By assumptions (1) and (2) and Remark A.14, all of the a_i are in the multiplicative domain of ϕ, i.e., the largest C*-subalgebra of $\mathrm{C}^*(X)$ for which ϕ is a *-homomorphism. By assumption (3), that means $\mathrm{C}^*(X)$ is the multiplicative domain of ϕ, or in other words, ϕ is a *-homomorphism. Uniqueness now follow from the fact that the *-homomorphism ϕ is determined by what it does on the generating set X. □

[47] Note that in this lemma there is a typo where it is written that the co-domain of ϕ is spanned by unitaries and not its range.

Corollary A.16. *Let $X \subseteq B(\mathcal{H})$ a be unital operator subspace and let $\phi : X \to B(\mathcal{H}')$ be a unital linear map.*

- *$\phi : X \to B(\mathcal{H}')$ is a complete isometry and $\phi(X)$ is spanned by unitary elements in $B(\mathcal{H}')$.*
- *$\phi : X \to B(\mathcal{H}')$ is completely contractive, X is the span of unitaries in $\mathrm{C}^*(X)$, and ϕ maps these to unitaries in $B(\mathcal{H}')$.*

Remark A.17 (On other linearization tricks). Pisier's trick is often used when you are working with a C*-algebra A with generators $\{W_i\}_i$ and an assignment $W_i \to V_i$ sending those generators to unitary generators of another C*-algebra B, and you want to know if the natural extension of this assignment to *-polynomials in the generators gives a *-homomorphism $A \to B$ when extend the assignment to polynomials. This trick reduces this problem to just checking whether the *linear* extension of this assignment to the operator space spanned by the $\{W_i\}_{i\in I}$ satisfies certain assumptions.

The trick is powerful, but the fact that one is dealing with unitaries is a strong assumption. Haagerup and Thorbjørnsen needed a version for self-adjoints instead. Still the overall theme of their linearization trick was to reduce a question about when questions about (*-)polynomials on a set of generators could be reduced to ones about linear polynomials on the generating set. Without unitaries, a result as straightforward as Pisier's is not possible, and so Haagerup and Thorbjørnsen instead reduced the questions to ones about linear polynomials on the generating set *with matrix coefficients*. In other words, (taking from a lecture of Speicher) a linearization trick is about

$$\begin{array}{rcl} \text{non-linear problem} & \longrightarrow & \text{(operator-valued) linear problem} \\ p(x_1, ..., x_n) & \longrightarrow & \alpha_0 \otimes 1 + \alpha_1 \otimes x_1 + \ldots + \alpha_n \otimes x_n \end{array}$$

B Tensor Products

This appendix completely ignores several interesting varieties of tensor products and useful facts.

B.1 For C*-algebras

Much of this section comes from chapter 3 from [7] and so arguments will not be given that can be found in that text. Deviations will be proved.

Let A, B, and C be C*-algebras and $\mathcal{H}$ a Hilbert space.

We denote an algebraic tensor product (with natural involution) by

$$A \odot B = \left\{ \sum_{i=1}^{n} a_i \otimes b_i : a_i \in A, b_i \in B, \text{ for } 1 \le i \le n, n \in \mathbb{N} \right\}.$$

To make this a C*-algebra, we would like to take its completion with respect to a C*-norm $\|\cdot\|_\alpha$ (which will actually be cross norms[48]). We denote the C*-algebra $\overline{A \odot B}^{\|\cdot\|_\alpha} = A \otimes_\alpha B$.

[48] $\|\cdot\|_\alpha$ is a cross norm if $\|a \otimes b\|_\alpha \le \|a\|_A \|b\|_b$ for any $a \in A$ and $b \in B$.

However, on this algebra, we can define multiple C*-norms $\|\cdot\|_\alpha$, which are often distinct. Indeed, we even have a name for C*-algebras whose algebraic tensor product with any other algebra has a unique C*-norm.

Definition B.1. A C*-algebra A is **nuclear** if $A \odot B$ has a unique C*-norm for any C*-algebra B.

Example B.2. A finite-dimensional C*-algebra is an easy and often useful example of a nuclear C*-algebra. Nuclearity follows from the fact that the algebraic tensor product is automatically closed in this case. A harder, yet equally important example is an abelian C*-algebra

There may exist many distinct C*-norms on $A \odot B$, we focus on the largest and smallest.

Definition B.3 (Tensor Product Norms)**.** For $x = \sum_{i=1}^n a_i \otimes b_i \in A \odot B$ (the algebraic tensor product)

- The max norm: $\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi : A \odot B \to B(\mathcal{H}_\pi)$ is an irred *-rep $\}$
- The spatial norm: $\|x\|_{\min} = \|\sum_{i=1}^n \pi_A(a_i) \otimes \pi_B(b_i)\|_{B(\mathcal{H}_A \otimes H_B)}$
  where (π_A, H_A) and (π_B, H_B) are **any**[49] faithful reps of A and B, resp.

The completions are $A \otimes_{\max} B = \overline{A \odot B}^{\|\cdot\|_{\max}}$ and $A \otimes_{\min} B = \overline{A \odot B}^{\|\cdot\|_{\min}}$.

Theorem B.4 (Universal Property of $\otimes_{\max}$)**.** *For any* C*-*algebra* C*, any *-homomorphism* $A \odot B \to C$ *extends uniquely to a *-homomorphism* $A \otimes_{\max} B \to C$*. In particular, if* $\pi_A : A \to C$ *and* $\pi_B : B \to C$ *are *-homomorphisms with commuting ranges, then the map* $\pi_A \times \pi_B$ *extends uniquely to* $A \otimes_{\max} B$*.*

Theorem B.5 (Restrictions of Tensor Product Maps)**.** *If* $\pi : A \odot B \to B(\mathcal{H})$ *is a *-homomorphism, then there exist restrictions*

$$\pi_A : A \to B(\mathcal{H}) \text{ and } \pi_B : B \to B(\mathcal{H})$$

with commuting ranges such that $\pi = \pi_A \times \pi_B$ *(where* $\pi_A \times \pi_B(a \otimes b) = \pi_A(a)\pi_B(b)$*).*
If A *and* B *are unital, we may replace* $B(\mathcal{H})$ *with any unital* C*-*algebra* C*.*

By the universality of $\|\cdot\|_{\max}$ and a theorem due to Takesaki (see [7, Chapter 3]), we know these are truly norms and that for any $x \in A \odot B$ and any C*-norm $\|\cdot\|_\alpha$ on $A \odot B$,

$$\|x\|_{\min} \leq \|x\|_\alpha \leq \|x\|_{\max}.$$

In other words, we have natural surjective *-homomorphisms

$$A \otimes_{\max} B \to A \otimes_\alpha B \to A \otimes_{\min} B.$$

Consequently, showing $A \odot B$ has a unique C*-norm is equivalent to showing that

$$A \otimes_{\min} B = A \otimes_{\max} B.$$

[49] See [7, Prop 3.3.11] for a proof.

Propostion B.6. *For any* C*-*algebras* A *and* B,

$$A \otimes_{\max} B \simeq B \otimes_{\max} A$$
$$A \otimes_{\min} B \simeq B \otimes_{\min} A$$

We conclude with the functoriality of $\otimes_{\max}$ and $\otimes_{\min}$.

Theorem B.7 (Continuity of Tensor Product Maps)**.** *If* $\phi : A \to C$ *and* $\psi : B \to D$ *are cp maps, then* $\phi \odot \psi : A \odot B \to C \odot D$ *extends to*

- *a cp map* $\phi \otimes_{\max} \psi : A \otimes_{\max} B \to C \otimes_{\max} D$ *and*
- *a cp map* $\phi \otimes_{\min} \psi : A \otimes_{\min} B \to C \otimes_{\min} D$.

Two key failures of $\otimes_{\max}$ and $\otimes_{\min}$ are the following:

1. For any C*-algebra C, while $\otimes_{\max} C$ preserves exact sequences (see e.g., [7, Proposition 3.7.1]), $\otimes_{\min} C$ does not always. (If it does for any exact sequence, then C is exact by definition.)
2. For any C*-algebra C, while $\otimes_{\min} C$ does preserve embeddings (which comes down to the fact that the spatial norm is independent of faithful representations), $\otimes_{\max} C$ does not always. (This will connect to WEP.)

Even though $\otimes_{\min} C$ does not necessarily preserve exact sequences, we still have the following useful fact.

Propostion B.8. *For* C*-*algebras* C *and* B *with* J *a closed two-sided ideal of* B, *there is a* C*-*norm* $\|\cdot\|_\alpha$ *on* $C \odot (B/J)$ *such that*

$$(B/J) \otimes_\alpha C \simeq (B \otimes_{\min} C)/(J \otimes_{\min} C).$$

Indeed, since the algebraic tensor product preserves exactness, we have an embedding

$$(B/J) \odot C \cong (B \odot C)/(J \odot C) \hookrightarrow (B \otimes_{\min} C)/(J \otimes_{\min} C),$$

and so we simply define α to be the completion there.

B.2 For Operator Systems/Spaces

A quick crash-course in tensor products for operator spaces can be found in [7, Appendix B]. For the original resource on tensor products of operator systems, we direct you to the development of these concepts in [28]. The theory guarantees that they play nicely and intuitively with the same notions for C*-algebras. For the sake of simplicity, we give pertinent characterizations for definitions.

Definition B.9. For two operator systems S_i with unital completely isometric embeddings $\iota_i : S_i \to B(\mathcal{H}_i)$, $S_1 \otimes_{\min} S_2$ is the operator system structure arising from the embedding $\iota_1 \odot \iota_2 : S_1 \odot S_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

In a similar fashion, we define a minimal tensor product of two operator subspaces $X_i \subset A_i$ of C*-algebras by inheriting the norm from $X_1 \odot X_2 \subset A_1 \otimes_{\min} A_2$ and completing in this norm.

Since the restriction of a faithful $*$-homomorphism on a C*-algebra A to an operator system S contained in A is unital completely isometric, we can conclude that for C*-algebras A_i with sub-operator systems (resp. spaces) $S_i \subseteq A_i$, there is a natural embedding $S_1 \otimes_{\min} S_2 \subseteq A_1 \otimes_{\min} A_2$. Hence, for $x \in S_1 \odot S_2$, we can say $\|x\|_{S_1 \otimes_{\min} S_2} = \|x\|_{A_1 \otimes_{\min} A_2}$.

Furthermore, if A and B are C*-algebras, then their operator system tensor product $A \otimes_{\min} B$ agrees with their C*-tensor product $A \otimes_{\min} B$.

Furthermore, the operator system/space tensor product $\otimes_{\min}$ is functorial, i.e.

Propostion B.10. *[7, Corollary 3.5.5] Given ucp (resp. cb) maps $\phi_i : S_i \to T_i$ between operator systems (resp. spaces), the map*

$$\phi_1 \odot \phi_2 : S_1 \odot S_2 \to T_1 \odot T_2$$

extends to a ucp (resp. cb) map

$$\phi_1 \otimes_{\min} \phi_2 : S_1 \otimes_{\min} S_2 \to T_1 \otimes_{\min} T_2$$

(which satisfies $\|\phi_1 \otimes_{\min} \phi_2\|_{cb} = \|\phi_1\|_{cb}\|\phi_2\|_{cb}$).

Remark B.11. This does not hold for $\otimes_{\max}$ ([22]).

B.3 Operator Space Duals

The notion of operator space duals comes from [5, 16]. There is a nice crash-course in this in [7, Appendix B]. To claim that two operator spaces X, Y are completely isometrically isomorphic, one must take their concrete C*-embeddings $X \subset B(\mathcal{H}_X)$ and $Y \subset B(\mathcal{H}_Y)$ and produce a map between the two that is completely isometric with respect to the norms inherited from the ambient C*-algebras (or their matrix amplifications). For $(X)^*$, the correct embedding is into

$$B\left(\bigoplus_{m\in\mathbb{N}} \bigoplus_{x\in M_n(\ell_n^\infty), \|x\|\leq 1} \ell_m^2\right).$$

But there is a nicer way to pick up on a dual operator system's norm structure. For an operator system X and each $n \in \mathbb{N}$, one considers the bijection from $M_n(X^*)$ to the space $\mathrm{CB}(X, M_n)$ of cb maps from X to M_n given by

$$[\varphi_{ij}]_{ij}^n \longleftrightarrow T_\varphi : x \mapsto [\varphi_{ij}(x)]_{ij}^n.$$

It turns out that with the aforementioned operator space embedding for X^*, this embedding is isometric for all n. Since the operator space structure *is* the norm on its matrix amplifications, we can alternatively define it's operator space structure by defining norms on $M_n(X^*)$

for each n via these embeddings. What's more, we can do it all at once and instead consider the embedding $X^* \odot \mathcal{K} \hookrightarrow \mathrm{CB}(X, \mathcal{K})$ given by sending each $z = \sum_k \varphi_k \otimes y_k \in X^* \odot \mathcal{K}$ to the finite-rank map $T_z : X \to \mathcal{K}$ given by

$$T_z(x) = \sum_k \varphi_k(x) y_k, \ \forall x \in X. \tag{B.1}$$

Remark B.12. By identifying $M_n(X^*)$ with $X^* \odot M_n = X^* \otimes M_n$, we see this generalized the map $M_n(X^*) \to \mathrm{CB}(X, M_n)$ above.

This embedding (or equivalently all such embeddings for M_n) is actually how Pisier defines the operator space dual in his book (see [43, Chapter 2.3]).

In fact, the operator space structure on X^* is such that the embedding $X^* \odot Y \to \mathrm{CB}(X, Y)$ defined just as in (B.1) is isometric for any operator space Y. On the flip side, suppose we have another operator space $\mathcal{S}$ inside a given C*-algebra A that we want to check is completely isometrically isomorphic to X^*. To do this, it suffices to produce a linear bijection $\psi : \mathcal{S} \to X^*$ so that $\psi \odot \mathrm{id}_{\mathcal{K}}$ composed with the embedding $X^* \odot \mathcal{K} \hookrightarrow \mathrm{CB}(X, \mathcal{K})$ remains isometric. Moreover, it equivalently suffices to do this with $\mathcal{K}$ replaced by a larger operator space, like $B(\ell^2)$.

C Residual Finite Dimensionality

Definition C.1. A C*-algebra A is *residually finite dimensional (RFD)* if it admits a separating family of finite-dimensional representations (i.e., for each $a \in A$, there exists n and $\pi : A \to M_n$ so that $\pi(a) \neq 0$). Equivalently, (by taking the direct sum of these representations) A embeds into some product $\prod_\kappa M_{n_k}$ of matrix algebras (countable if A is separable). Equivalently, for any $a \in A$, $\|a\| = \sup\{\|\pi(a)\| \mid \pi \text{ a finite dimensional rep}\}$.

Remark C.2. This likely sounds like residual finiteness for groups (separating family of finite representations). In fact a finitely generated amenable group has an RFD full/reduced C*-algebra iff it is RF ([3, 35]). For non-amenable groups, it's less clear. While Choi proved that $\mathrm{C}^*(\mathbb{F}_n)$ is RFD ([9]), Bekka has proved that many RF groups, like $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, do not have RFD full group C*-algebras.[50]

Propostion C.3. *If A and B are RFD* C*-*algebras, then $A \odot B$ has a unique* C*-*norm iff $A \otimes_{\max} B$ is RFD.*

Proof. The ($\Rightarrow$) claim follows immediately from the fact that $A \otimes_{\min} B$ is RFD when A and B are RFD. (Indeed, if $A \otimes_{\min} B$ is RFD, then so are A and B as subalgebras; on the other hand, if A and B are RFD, with separating families of f.d. representations $\{\pi_\alpha\}$ and $\{\sigma_\beta\}$, then $\pi = \oplus_\alpha \pi_\alpha$ and $\sigma = \oplus_\beta \sigma_\beta$ are faithful representations. Hence, $\{\pi_\alpha \otimes \sigma_\beta\}_{\alpha,\beta}$ yields a separating family of finite dimensional representations for $A \otimes_{\min} B$.)

For the ($\Leftarrow$) we show that any finite-dimensional representation $\pi : A \otimes_{\max} B \to M_n$ factors through $A \otimes_{\min} B$:

[50] As non-nuclear reduced group C*-algebras tend to have far fewer finite-dimensional representations, the question is yet again primarily interesting in the full setting.

$$\begin{tikzcd}
A \otimes_{\max} B \arrow[rr, "\pi"] \arrow[dr, "q"'] & & M_n \\
& A \otimes_{\min} B \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

To that end, let $\pi : A \otimes_{\max} B \to M_n$ be a finite dimensional representation with restrictions π_A and π_B. Since $\pi_A(A)$ and $\pi_B(B)$ are commuting C*-algebras, the natural inclusions $\iota_A : \pi_A(A) \to M_n$ and $\iota_B : \pi_B(B) \to M_n$ are representations with commuting ranges, and the universality of $\otimes_{\max}$ yields a representation

$$(\iota_A \times \iota_B) : \pi_A(A) \otimes_{\max} \pi_B(B) \to M_n$$

with $\pi_A(a) \otimes \pi_B(b) \mapsto \pi_A(a)\pi_B(b)$ for all $a \in A$ and $b \in B$.

On the other hand, π_A and π_B induce a representation

$$\pi_A \otimes \pi_B : A \otimes_{\min} B \to \pi_A(A) \otimes_{\min} \pi_B(B).$$

Now, since $\pi_A(A) \otimes_{\max} \pi_B(B) = \pi_A(A) \otimes_{\min} \pi_B(B)$, we may define the representation

$$\sigma := (\iota_A \times \iota_B) \circ (\pi_A \otimes \pi_B) : A \otimes_{\min} B \to M_n$$

of $A \otimes_{\min} B$ where for $a \otimes b \in A \odot B$,

$$\sigma(a \otimes b) = (\iota_A \times \iota_B)(\pi_A(a) \otimes \pi_B(b)) = \pi_A(a)\pi_B(b) = \pi(a \otimes b).$$

Then by linearity and continuity, $\pi = \sigma \circ q$. $\square$

More compactly, for a finite-dimensional representation $\pi : A \otimes_{\max} B \to M_n$ the following diagram commutes

$$\begin{tikzcd}
A \otimes_{\max} B \arrow[r, "\pi"] \arrow[d, "q"'] & M_n \\
A \otimes_{\min} B \arrow[r, "\pi_A \otimes \pi_B"] & \pi_A(A) \otimes \pi_B(B) \arrow[u, "\iota_A \times \iota_B"']
\end{tikzcd}$$

D Odds and Ends on von Neumann algebras

Here I will provide some relevant von Neumann algebraic definitions and facts used in this document. Unless useful for the arguments above, I will not provide extra context, related facts, equivalent characterizations, general cases, etc..

Remark D.1. A von Neumann algebra M is called *separably acting* if it can be faithfully represented as operators on a separable Hilbert space. Equivalently, one can say it has a separable pre-dual.

Definition D.2. A separably acting von Neumann algebra is of type II_1 if it is infinite dimensional and has a faithful normal trace[51].

A von Neumann algebra is a *factor* if it has trivial center.[52]

[51] This is a linear functional $\tau : M \to [0, \infty)$ which satisfies the trace condition ($\tau(ab) = \tau(ba)$ for all a, b), is faithful ($\tau(x^*x) = 0 \Rightarrow x = 0$), normal ($\tau(\sup x_i) = \sup \tau(x_i)$), and finite ($\tau(1) < \infty$)

[52] Think of this as a simplicity criteria.

Remark D.3. For a separably acting infinite dimensional von Neumann algebra, being finite is the same as being tracial.

Definition D.4. [46, Theorem 2.15] A von Neumann algebra M is *semifinite* iff it has a faithful normal extended trace $\tau : M_+ \to [0, \infty]$ such that for every $x \in M_+$, there exists a $y \in M_+$ with finite trace such that $y \leq x$.

Propostion D.5. *Any semifinite von Neumann algebra $M \subseteq B(\mathcal{H})$ can be written as $\overline{\bigcup_i M_i}''$ where $\{M_i\}_i$ is an increasing net of finite (tracial) unital von Neumann subalgebras.*

Proof. Let τ be a faithful semifinite trace on M and let $(p_i)_i$ be a family of mutually orthogonal projections with finite trace which is maximal in that sense. We claim $\sum_i p_i = 1$. If not, then there exists a nonzero $y \in M_+$ with finite trace such that $y \leq 1 - \sum_i p_i$. Then there exists some nonzero projection $q \in M$ and $\lambda > 0$ so that $y \geq \lambda q$, but this q contradicts the maximality of our collection. By taking larger and larger finite sums of these projections, we can produce a directed increasing net $(q_\lambda)_\lambda$ of projections with finite traces which converge weak* to 1. For each λ, $\tau|_{p_\lambda M p_\lambda}$ is now a finite faithful normal trace. Then the unital subalgebras $M_\lambda := p_\lambda M p_\lambda + \mathbb{C}(1 - p_\lambda)$ are finite with weak*-dense union in M. $\square$

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